Characterisations for the category of Hilbert spaces

By

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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Abstract

Category theory is an algebraic framework based on the composition of functions. Categories consist of objects and morphisms between objects. A dagger category is a type of category which has a notion of reversibility for each morphism. A monoidal category is one which allows the joining of objects and of morphisms in parallel, rather than in series as with composition. This joining is done in such a way as to satisfy certain coherence conditions.

The categories of real and of complex Hilbert spaces with bounded linear maps are dagger monoidal categories and have received purely categorical characterisations by Chris Heunen and Andre Kornell. This characterisation is achieved through Solèr's theorem, a result which shows that certain orthomodularity conditions on a Hermitian space over an involutive division ring result in a Hilbert space with the division ring being either the reals, complexes or quarternions.

The Heunen-Kornell characterisation makes use of a monoidal structure, which in turn excludes the category of quarternionic Hilbert spaces. We provide an alternative characterisation without the assumption of monoidal structure on the category. This new approach not only characterises the categories of real and of complex Hilbert spaces, but also the category of quaternionic Hilbert spaces.

Contents

The world is all that is the case.

Ludwig Wittgenstein

1 Introduction

The theory of Hilbert space is deeply geometric. Euclidean space; a complete finite dimensional space commonly used to represent a state and phase space in classical physics, facilitates the classical notions of length and angle. Hilbert space extends these notions to an infinite dimensional setting while retaining the notion of completeness.

The mathematical formalism of Hilbert space for quantum theory was developed by John von Neumann in 1932 with his seminal work *Mathematische Grundlagen der Quantenmechanik* (The Mathematical Foundations of Quantum Mechanics) [21], and in 1936, Garrett Birkhoff and John von Neumann showed that each physical system can be characterised by its property lattice, which is a complete, atomistic othocomplemented lattice that arises from the primitive notions of state, property and observable. In fact, the property lattice of a physical system which reflected the physical qualities of the system was found to be isomorphic to the orthomodular lattice of closed subspaces of a Hilbert Space, emphasising the importance of orthogonal projections onto linear subspaces [2]. In 1995 Maria Pia Solèr proved a remarkable theorem [20] which characterises Hilbert spaces using orthomodular spaces: Hermitian spaces which can be linearly decomposed into the orthogonal projections to a closed subspace and its complement. Solèr uses an infinite dimensional orthomodular space over an involutive division ring and states that if such a space contains an infinite orthonormal system then that division ring is necessarily the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$ with the associated orthomodular space being a Hilbert space over the respective scalars. In 2022, Chris Heunen and Andre Kornell use Solèr's theorem in their noteworthy paper *Axioms for the category of Hilbert spaces* [9] to characterise the category of K-Hilbert spaces and bounded K-linear maps, $\mathbf{Hilb}_K$, in purely categorical terms for K equal to $\mathbb{R}$ or $\mathbb{C}$. Axioms (T) on page 1 [9] introduces a monoidal structure on the category which forces the scalars K to be commutative and hence excludes the possibility for K to be the quaternions.

In this thesis we will explore the categorical structure required to characterise $\mathbf{Hilb}_K$

for K equal to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. This means we have to create a suitable list of conditions on a category $\mathbf{C}$ to establish an equivalence with $\mathbf{Hilb}_K$. To achieve this we exclude the assumption of a monoidal structure and alter the associated unit to instead be a chosen object which is a simple generator rather than a simple monoidal generator and make this an axiom to replace (T). The rest of our axioms resemble all but (T) and thus many of the arguments towards the equivalence resemble those found in [9] except where the monoidal structure comes into effect and in particular when the commutativity of the scalars is used.

In Chapter 2 we will briefly introduce relevant algebraic structures from monoids to modules with their respective categories and then introduce Hermitian and Hilbert spaces. A structural view of the division rings $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ is presented with the purpose to understand familiar theorems from functional analysis in the real and complex setting to then be extended to apply to the quaternionic case.

In Chapter 3 we begin building the structure on a locally small category $\mathbf{C}$ with a chosen object G by equipping the category with a biproduct $\oplus : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$. This is the first step towards a notion of linear structure on the hom-sets when we define a compositionally defined scalar multiplication $\cdot : \mathbf{C}(G, H) \times \mathbf{C}(G, G) \to \mathbf{C}(G, H)$. This motivates directing attention to $\mathbf{C}(G, G)$ which will become the base algebra for a semimodule $\mathbf{C}(G, H)$ for some object H of $\mathbf{C}$.

In Chapter 4 we look at dagger category theory and work through the preliminary propositions and lemmas which will be used throughout. We investigate the effect of the dagger structure on hom-sets of the form $\mathbf{C}(G, H)$ and eventually show this hom-set to be a Hermitian space with a dagger defined Hermitian form and scalars $\mathbf{C}(G, G)$. Much of this section is a study of work from [7–10, 18, 19].

In Chapter 5 we show that the system of dagger subobjects of H denoted $\mathrm{Sub}_\dagger(H)$ is a complete ortholattice and is isomorphic to the ortholattice of closed subspaces of $\mathbf{C}(G, H)$. Completeness of $\mathrm{Sub}_\dagger(H)$ requires the wide subcategory $\mathbf{C}_{\mathrm{dm}}$ consisting of the dagger monomorphisms to have directed colimits.

Our main result appears in Chapter 6, where we show that $\mathbf{C}(G, H)$ is a Hilbert space with scalars $\mathbf{C}(G, G)$ isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Finally, we show a dagger equivalence between $\mathbf{C}$ and $\mathbf{Hilb}_{\mathbf{C}(G,G)}$.

In Chapter 7 we introduce dagger monoidal structure and compare our characterisation with the characterisation found in [9].

2

Algebraic Structures and Hilbert Space

This is mostly a preliminary introduction to a few of the algebraic structures we will encounter in the coming chapters as well as Hilbert spaces and their relevant theorems. It is important to note that while the structure of a Hilbert space may seem like a purely analytical construction we will make use of Solèr's theorem, a miraculous result that allows for the characterisation of real, complex or quaternionic Hilbert spaces in a purely algebraic sense. The division rings $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ are of special interest for this project and it will be essential to understand how inner product spaces over either $\mathbb{C}$ or $\mathbb{H}$ can be equivalent to a real inner product space equipped with additional structure.

2.1 Categories

Definition 2.1.1 (Category)**.** A *category* $\mathbf{C}$ consists of:

- A collection of *objects* denoted ob$\mathbf{C}$.
- For each $A, B \in \text{ob}\mathbf{C}$, a collection of *morphisms* $\mathbf{C}(A, B)$. Sometimes morphisms are also referred to as *maps* or *arrows* and diagrammatically written as,

$$A \xrightarrow{f} B$$

- For each $A, B, C \in \text{ob}\mathbf{C}$, a composition operation

$$\begin{aligned} \circ : \mathbf{C}(B, C) \times \mathbf{C}(A, B) &\to \mathbf{C}(A, C) \\ (g, f) &\mapsto g \circ f \end{aligned}$$

  and diagrammatically written as commuting diagrams,

$$\begin{tikzcd} A \arrow[rr, "g \circ f"] \arrow[dr, "f"'] & & C \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

- This composition is associative which is to say it is subject to the condition that

$$(h \circ g) \circ f = h \circ (g \circ f).$$

- For each $A \in \text{ob}\mathbf{C}$, an *identity* morphism $\text{id}_A : A \to A$ (and sometimes written 1_A or 1 when the context is understood) where for each $f \in \mathbf{C}(A, B)$ and $g \in \mathbf{C}(B, A)$, $f \circ \text{id}_A = f$ and $\text{id}_A \circ g = g$.

Definition 2.1.2 (Subcategory)**.** A *subcategory* $\mathbf{D}$ of a category $\mathbf{C}$ is a category which has $\text{ob}\mathbf{D} \subseteq \text{ob}\mathbf{C}$ and $\mathbf{D}(A, B) \subseteq \mathbf{C}(A, B)$ for each $A, B \in \text{ob}\mathbf{D}$, with composition and identities as defined in $\mathbf{C}$. If $\text{ob}\mathbf{D} = \text{ob}\mathbf{C}$ then $\mathbf{D}$ is called a *wide subcategory* of $\mathbf{C}$. If $\mathbf{D}(A, B) = \mathbf{C}(A, B)$ for each $A, B \in \text{ob}\mathbf{D}$, then $\mathbf{D}$ is called a *full subcategory* of $\mathbf{C}$.

If for each object A, B in $\mathbf{C}$ the collection of morphisms $\mathbf{C}(A, B)$ forms a set, we say that $\mathbf{C}$ is *locally small* and that $\mathbf{C}(A, B)$ is the *hom-set* of A and B. An important example of a locally small category is the category of sets and set-functions, denoted **Set**. Since all of the categories mentioned in this text are subcategories of **Set**, we will assume categories are locally small.

Many algebraic structures sit in their own category which consists of all such algebraic structures and the structure preserving maps between them. The following is a list of some of the relevant algebraic structures as well as their corresponding categories.

A *monoid* $(M, \cdot, 1)$ consists of a set M with an element 1 called the identity together with a composition/multiplication operation $\cdot : M \times M \to M : (a, b) \to a \cdot b$ where for each $a, b, c \in M$,

- $(ab)c = a(bc)$
- $1a = a1 = a$

Given two monoids $(M, \cdot, 1_M)$ and $(N, \cdot, 1_N)$, a *monoid homomorphism* is a function $f : M \to N$ which preserves the monoid structure in the sense that for each $a, b \in M$,

- $f(ab) = f(a)f(b)$
- $f(1_M) = 1_N$

A *commutative monoid* is a monoid where $ab = ba$ for each $a, b \in M$. The category of monoids and monoid homomorphisms is denoted **Mon** and the category of commutative monoids is denoted **CMon** and is a full subcategory of **Mon**.

Example 2.1.3. For any object $A \in \text{ob}\mathbf{C}$ in a category $\mathbf{C}$, the hom-sets $\mathbf{C}(A, A)$ form a monoid under composition with the monoid identity being id_A.

A *group* is a monoid $(G, \cdot, 1)$ where for each $a \in G$ there exists an inverse a^{-1} for which $aa^{-1} = a^{-1}a = 1$. When this monoid is commutative, G is called an *abelian group*. A *group homomorphism* is a function $f : G \to H$ between groups G and H which preserves the monoid structure, and therefore also preserves inverses in that $f(a^{-1}) = f(a)^{-1}$ for each $a \in G$. The category of groups and group homomorphisms is denoted **Grp** and is a full subcategory of **Mon**. The category of Abelian groups is denoted **Ab** and is a full subcategory of both **CMon** and **Grp**. The following diagram shows the full subcategory inclusion for some of the categories mentioned so far:

$$\begin{tikzcd} & \mathbf{Mon} & \\ \mathbf{CMon} \arrow[ur] & & \mathbf{Grp} \arrow[ul] \\ & \mathbf{Ab} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

A *semiring* $(R, +, \cdot, 0, 1)$ consists of a commutative monoid $(R, +, 0)$ with respect to an addition operation $+ : R \times R \to R$ and a monoid $(R, \cdot, 1)$ with respect to a multiplication operation $\cdot : R \times R \to R$. These addition and multiplication operations satisfy the left and right distributivity conditions for each $a, b, c \in R$,

- $a(b + c) = ab + ac$
- $(a + b)c = ac + bc$

A *semiring homomorphism* is a function $f : R \to S$ between two semirings R and S which preserves the $+$ and $\cdot$ monoid structures. The category of semirings and semiring homomorphisms is denoted **Rig**, with the word *rig* refering to the word *ring* without the letter 'n', signifying the exclusion of negation[1].

Example 2.1.4. In the next chapter we will discuss a categorical structure called a biproduct and prove the following result: For any object A in a category $\mathbf{C}$ with biproducts, the hom-set $\mathbf{C}(A, A)$ becomes a semiring with multiplication structure given by composition and biproduct defining addition.

A *ring* is a semiring $(R, +, \cdot, 0, 1)$ for which $(R, +, 0)$ is an abelian group. A *commutative ring* is a ring for which $(R, \cdot, 1)$ is a commutative monoid. A *ring homomorphism* is a function $f : R \to S$ between two rings which preserves the monoid and group structure of the addition and multiplication respectively. The category of rings and ring homomorphisms is denoted **Ring** and is a full subcategory of **Rig**. The category of commutative rings is denoted **CRing** and is a full subcategory of **Ring**.

A *division ring*[2] is a ring $(R, +, \cdot, 0, 1)$ with $(R\backslash\{0\}, \cdot, 1)$ as a group. The category of division rings is denoted **DRing** and is a full subcategory of **Ring**.

A *field* is a division ring $(K, +, \cdot, 1, 0)$ which is commutative in the sense that $(K, \cdot, 1)$ is a commutative monoid. The category of fields and field homomorphisms is denoted **Field** and is a full subcategory of both **CRing** and **DRing**.

An *involution* on a rig K is an operation $* : K \to K$ such that for each $a, b \in K$, $(a^*)^* = a$, $(a + b)^* = a^* + b^*$ and $(a \cdot b)^* = b^* \cdot a^*$.

Example 2.1.5.
- The field of real numbers $\mathbb{R}$ is a division ring with involution being the identity operation so that for any $a \in \mathbb{R}$ we have $a^* = a$.

[1] While some authors use 'rig' only in the commutative case, we will include the non-commutative case in our definition.

[2] A division ring is sometimes called a skew field as in [20].

- The field of complex numbers $\mathbb{C}$ is a division ring with involution being complex conjugation where for each $z \in \mathbb{C}$ we have $z^* = \bar{z}$. Complex numbers may be presented using pairs of real numbers so that a complex number $z = a + ib$ for $a, b \in \mathbb{R}$ and imaginary unit i satsifying $i^2 = -1$. Complex conjugation is then $\overline{a + ib} = a - ib$.

- The division ring of quaternions $\mathbb{H}$ is a division ring with involution being quaternionic conjugation where for $q \in \mathbb{H}$ we have $q^* = \bar{q}$. Similar to complex numbers, quaternionic numbers may be presented using quadruples of real numbers $a, b, c, d \in \mathbb{R}$ with a quaternion $q = a + bi + cj + dk$ and imaginary units i, j, k saysifying $i^2 = j^2 = k^2 = -1$ and $ij = k$, $jk = i$ and $ki = j$. Conjugation is then $\overline{a + bi + cj + dk} = a - bi - cj - dk$.

The following diagram shows this full subcategory inclusion.

$$
\begin{tikzcd}
 & \mathbf{Rig} & \\
 & \mathbf{Ring} \arrow[u] & \\
\mathbf{CRing} \arrow[ur] & & \mathbf{DRing} \arrow[ul] \\
 & \mathbf{Field} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

2.2 Modules

Definition 2.2.1. (Right Semimodule) Let K be a semiring. A *right K-semimodule* is a commutative monoid $(M, +, 0)$ together with a scalar multiplication $\cdot : M \times K \to M$ where for each $\lambda, \mu \in K$ and $u, v \in M$,

- *additive distributivity*: $(u + v) \cdot \lambda = u \cdot \lambda + v \cdot \lambda$

- *scalar distributivity*: $v \cdot (\lambda + \mu) = v \cdot \lambda + v \cdot \mu$

- *associativity of scalar multiplication*: $(v \cdot \lambda) \cdot \mu = v \cdot (\lambda\mu)$

- *scalar identity:* $v = v \cdot 1$

Example 2.2.2. The hom-set $\mathbf{C}(A, B)$ for any $A, B \in \mathrm{ob}\mathbf{C}$ of a category $\mathbf{C}$ with biproducts is a semimodule over the semiring $\mathbf{C}(A, A)$, a result that is explored in Chapter 3.

Definition 2.2.3. (Right Module) A right K-semimodule $(M, K, +, \cdot, 0)$ is called a *right K-module* when $(M, +, 0)$ is an abelian group. When K is a division ring, a right K-module is usually called a *right K-vector space*. The division ring K of a right K-vector space M we refer to as the *scalars* and the elements of M as *vectors*. If K has commutative multiplication we often drop the word right and speak simply of K-modules or K-vector spaces

Example 2.2.4. Any right semimodule $(M, +, \cdot, 0_M)$ over a ring $(R, +, \cdot, 0_M, 1)$ is a module since $-1 \in R$ and for each $u \in M$ we have $u \cdot (-1) \in M$ and so $u \cdot 1 + u \cdot (-1) = u \cdot (1 + (-1)) = u \cdot 0_R = 0_M$ and so u has an inverse $-u = u \cdot (-1)$ making $(M, +, 0_M)$ an abelian group.

Definition 2.2.5 (Module Homomorphism)**.** A *module homomorphism* $T : U \to V$ between right K-modules U and V is a function where for each $u, v \in U$ and $\lambda \in K$ we have,

- $T(u + v) = Tu + Tv$
- $T(u \cdot \lambda) = Tu \cdot \lambda$

When U and V are right K-vector spaces we call $T : U \to V$ a *K-linear map* or just *linear map* when K is understood.

Definition 2.2.6 (Direct Sum)**.** A *direct sum* of two right K-modules H and K is defined as,

$$H \oplus K := \{(h, k) \mid h \in H, k \in K\}$$

with the expected addition and scalar multiplication. The category of right K-modules and module homomorphisms is denoted $\mathbf{Mod}_K$. When K is a division ring we may write $\mathbf{Vect}_K$ instead of $\mathbf{Mod}_K$.

Example 2.2.7.

- Previously we said that complex numbers can be presented as combinations of real numbers with an imaginary unit i. A similar notion applies to real and complex vector spaces. A complex vector space is equivalent to a real vector space V equipped with a linear map $s : V \to V$ satisfying $s^2 = -1$. The scalar multiplication by i is given by the action of s. In this way s corresponds to i. Linear maps on complex right vector spaces characterised in this way are the linear maps on a real vector space V that commutes with s.
- Similarly, a quaternionic right vector space is equivalent to a real vector space V together with linear maps $s, t : V \to V$ that satisfy $s^2 = t^2 = -1$ and $ts = -st$. Like before, these linear maps act on V as the quaternionic imaginary units would in the sense that s corresponds to i, t to j and ts to k. Linear maps on quaternionic right vector spaces characterised in this way are the linear maps on a real vector space V that commute with s and t.

Definition 2.2.8 (Hermitian Form)**.** Let K be a division ring with involution. For a right K-vector space V, a *Hermitian form* on V is an operation $\langle \cdot, \cdot \rangle : V \times V \to K$ which takes a pair (u, v) to an element $\langle u, v \rangle$ in K and satisfies the following,

1. $\forall u, v \in V$, $\langle u \cdot \lambda, v \rangle = \langle u, v \rangle \lambda$ and $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
2. $\forall u, v \in V$, $\langle u, v \rangle = \langle v, u \rangle^*$

A right K-vector space with a Hermitian form is called a *right K-Hermitian space.* A Hermitian form is *nonsingular* if,

- If $\forall v \in V$, $\langle a, v \rangle = 0$ or equivalently $\forall v \in V$, $\langle v, a \rangle = 0$, then $a = 0$

Remark 2.2.9. Antilinearity in the second argument follows from conditions 1 and 2 so that also

3. $\forall u, v \in V$, $\langle u, v \cdot \lambda \rangle = \lambda^* \langle u, v \rangle$ and $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$.

Definition 2.2.10 (Orthogonality)**.** Let $(U, \langle \cdot, \cdot \rangle)$ be a Hermitian space. Two vectors $u, v \in U$ are said to be *orthogonal*, written $u \perp v$, when $\langle u, v \rangle = 0$. Let V, W be subspaces of U; then V is orthogonal to W, written $V \perp W$, when $v \perp w$ for each $v \in V$ and $w \in W$. The *orthogonal complement* to a subspace $V \subseteq U$ is the subspace $V^\perp \subseteq U$ such that,

$$V^\perp := \{u \in U \mid \forall v \in V, u \perp v\}$$

Definition 2.2.11 (Closed Subspace)**.** A subspace F of a Hermitian space V is said to be *closed* when $(F^\perp)^\perp = F$.

Definition 2.2.12 (Direct Sum)**.** The *direct sum* of two right Hermitian spaces H and K is defined as their direct sum $H \oplus K$ as modules with the Hermitian form defined for each $(h_1, k_1), (h_2, k_2) \in H \oplus K$ as,

$$\langle (h_1, k_1), (h_2, k_2) \rangle := \langle h_1, h_2 \rangle_H + \langle k_1, k_2 \rangle_K$$

Definition 2.2.13 (Orthomodular Space)**.** A Hermitian space V is called *orthomodular* if for any closed subspace $F \subseteq H$ we have $H = F \oplus F^\perp$.

Definition 2.2.14 (Orthonormal system)**.** A set of vectors $(e_i)_{i \in I} \subset H$ in a Hermitian space V with index set I is said to be an *orthogonal system* if $\langle e_i, e_j \rangle = 0$ for each $i, j \in I$ when $i \neq j$, and is *orthonormal* when moreover $\langle e_i, e_i \rangle = 1$ for each $i \in I$.

2.3 Hilbert Spaces

Definition 2.3.1 (Inner Product)**.** Let $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ with the usual involutions. An *inner product* on a right K-vector space V is a Hermitian form $\langle \cdot, \cdot \rangle : V \times V \to K$ that is *positive definite* in that,

- $\langle v, v \rangle \in \mathbb{R}$, $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$.

A right K-vector space with an inner product is called an *inner product space*.

Definition 2.3.2 (Norm)**.** Let $\langle \cdot, \cdot \rangle : V \times V \to K$ be an inner product, we call the map $\| \cdot \| : V \to \mathbb{R}$ defined by $\|v\| = \langle v, v \rangle^{1/2}$ the *induced norm*. The norm of a linear map $T : U \to V$ is defined as

$$\|T\| := \sup\{\|Tx\| : \forall x \in U, \; \|x\| \leq 1\}$$

Definition 2.3.3 (Bounded Linear Map)**.** Let $T : H \to K$ be a linear map between inner product spaces U and V. T is *bounded* in the induced norm when there exists an $M \in \mathbb{R}$ such that,

$$\|T\| \leq M$$

Definition 2.3.4 (Adjoint)**.** Let $f : U \to V$ be a bounded linear map between two right inner product spaces $(U, \langle\cdot,\cdot\rangle_U)$ and $(V, \langle\cdot,\cdot\rangle_V)$. The *adjoint* to f, if it exists, is a linear map $f^\dagger : V \to U$ where for each $v \in V$ and $u \in U$,

$$\langle f^\dagger(v), u\rangle_U = \langle v, f(u)\rangle_V$$

A linear map $f : U \to U$ is called *self adjoint* when $f = f^\dagger$ and *unitary* when f is invertible and for each $u, u' \in U$,

$$\langle f(u), f(u')\rangle_V = \langle u, u'\rangle_U$$

Example 2.3.5. (i) Continuing with our familiar examples, a real inner product space V with inner product $[\cdot,\cdot] : V \times V \to \mathbb{R}$ together with a linear map $s : V \to V$ such that $s^2 = -1$ and $s^\dagger = -s$, with respect to $[\cdot,\cdot]$, is equivalent to a complex inner product space with the inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}} : V \times V \to \mathbb{C}$ defined as,

$$\langle u, v\rangle_{\mathbb{C}} := [u, v] - [su, v]i$$

- (ii) Similarly, a real inner product space V with inner product $[\cdot,\cdot] : V \times V \to \mathbb{R}$ together with linear maps $s, t : V \to V$ such that $s^2 = t^2 = -1$, $ts = -st$, $s^\dagger = -s$ and $t^\dagger = -t$ is equivalent to a quaternionic inner product space with the inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}} : V \times V \to \mathbb{H}$ defined as,

$$\langle u, v\rangle_{\mathbb{H}} := [u, v] - [su, v]i - [tu, v]j - [tsu, v]k$$

Lemma 2.3.6. *Let $s, t : V \to V$ be the linear maps from the example above. For any $u \in V$, we have $u \perp su$, $u \perp tu$ and $u \perp tsu$.*

Proof. It follows from the symmetry of the real inner product that,

$$\begin{aligned} [sx, x] &= [x, s^\dagger x] && \text{(by adjoint)} \\ &= [x, -sx] && \text{(since } s^\dagger = -s) \\ &= -[x, sx] && \text{(by linearity)} \\ &= -[sx, x] && \text{(by symmetry)} \end{aligned}$$

and so $[sx, x] = 0$. A similar argument can be made for t. Now,

$$[tsu, u] = [-stu, u] = [tu, su] = [u, -tsu] = -[u, tsu] = -[tsu, u]$$

and hence $[tsu, u] = 0$. □

Theorem 2.3.7. *Let $K = \mathbb{C}$ or $\mathbb{H}$ and $(V, [\cdot,\cdot])$ be a K-inner product space. Then for each $x \in V$,*

$$[x, x] = \langle x, x\rangle_{\mathbb{C}} = \langle x, x\rangle_{\mathbb{H}}$$

Proof. We will only show the quaternionic case. It follows directly from the above lemma that for each $x \in V$,

$$\begin{aligned} \langle x, x\rangle_{\mathbb{H}} &= [x, x] - [sx, x]i - [tx, x]j - [tsx, x]k \\ &= [x, x] - 0i - 0j - 0k \\ &= [x, x] \end{aligned}$$

□

This is a very convenient theorem and means that any statements about boundedness that apply to real inner product spaces can also apply to complex or quaternionic inner product spaces. This includes statements about completeness as in the following definition.

Definition 2.3.8 (Right Hilbert Space)**.** Let $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. A *right K-Hilbert space* $(H, \langle \cdot, \cdot \rangle : H \times H \to K)$ is a right inner product space that is complete in the induced norm $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$, which is to say that all Cauchy sequences of H converge in H.

An equivalent statement for completeness is to say that if a sequence $\{x_n\}_{n=1}^{\infty}$ of vectors in H satisfies $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then there exists a vector $x \in H$ such that $\|x - \sum_{n=1}^{N} x_n\| \to 0$ as $N \to \infty$. In other words, every absolutely convergent series converges.

Theorem 2.3.9. *Let $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $T : H \to H'$ be a linear map between K-Hilbert spaces H and H'. If T has an adjoint $T^{\dagger}$ then T is bounded.*

Proof. Let (x_n) be a sequence in H such that $x_n \to x$ and $Tx_n \to z$ when $n \to \infty$. Then for each $y \in K$, $\langle Tx_n, y \rangle \to \langle z, y \rangle$ but also $\langle Tx_n, y \rangle = \langle x_n, T^{\dagger} y \rangle \to \langle x, T^{\dagger} y \rangle = \langle Tx, y \rangle$. It follows from the uniqueness of limits that $Tx = z$. By theorem 4.13-3 in [14], T is a closed linear operator and then by the closed graph theorem 4.13-2 in [14] T is bounded. Theorem 2.3.7 guarantees this holds for $\mathbb{H}$. $\square$

The category of K-Hilbert spaces and bounded linear maps is denoted $\mathbf{Hilb}_K$ and is the star[3] category of this thesis.

Definition 2.3.10 (Basis)**.** Let H be a right K-Hilbert space and let $E := \{e_i\}_{i \in I}$ be a family of elements of H, indexed by a set I. The family E is called an *orthogonal basis* of H if it satisfies the following conditions:

- E is an orthogonal system.
- If $I = \mathbb{N}$ and $a \in H$, then there exists a family of coefficients $\{a_i \in K \mid i \in \mathbb{N}\}$ such that $\|a - \sum_{i=1}^{n} a_i e_i\| \to 0$ as $n \to \infty$.

If E is also an orthonormal system we say that E is an *orthonormal basis.*

Theorem 2.3.11 (Page 168 [14])**.** *Each Hilbert space has an orthonormal basis.*

Definition 2.3.12 (Dimension)**.** The *dimension* of a Hilbert space H is the cardinality of an orthonormal basis of H.

The dimension of a Hilbert space is well defined by the fact that any two orthonormal bases of the same Hilbert space have the same cardinality.

Theorem 2.3.13 (Theorem 3.6-5 [14])**.** *Let $K = \mathbb{R}$ or $\mathbb{C}$. Two right K-Hilbert spaces H and $\tilde{H}$ are isomorphic if and only if they have the same dimension.*

Theorem 2.3.14. *Let $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Any bounded K-linear map $T : H \to H$ on an infinite dimensional K-Hilbert space H can be decomposed into a linear combination of unitary maps.*

[3] Categories such as **Hilb** are dagger categories, discussed in chapter 3. The term dagger referres to the † notation in physics signifying the adjoint. Historically the $*$ symbol has been to signify such operations and categories such as **Hilb** were referred to as *star* categories.

Proof. See appendix A □

In 1995, Maria Pia Solèr developed a characterisation of Hilbert spaces using orthomodular spaces. This is a theorem that is inspired by results in quantum logic by Birkhoff and von Neumann in the mid 1930s [2], and built on the work by Kaplansky on infinite dimensional quadratic forms in the 1950s [11] and Piron's representation theorem which proved a correspondence between propositional systems and orthomodular spaces in the 1960s [16]. Solèr's theorem is originally stated as follows:

Theorem 2.3.15 (Solèr's Theorem [20])**.** *Let $(E, \langle\cdot,\cdot\rangle)$ be an infinite dimensional orthomodular space over a division ring K which contains an orthonormal system $(e_i)_{i\in\mathbb{N}}$. Then K is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and $(E, \langle\cdot,\cdot\rangle)$ is a Hilbert space over K.*

The categorical characterisation of $\mathbf{Hilb}_K$ for $K = \mathbb{R}, \mathbb{C}$ by Heunen and Kornell [9] uses Solèr's theorem to build an equivalence of categories. We will use Solèr's theorem in the same way for our characterisation of $\mathbf{Hilb}_K$ for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

3

Biproducts

In this chapter we introduce a way to construct objects from existing objects in a category $\mathbf{C}$ using a type of limit called a biproduct. We will then explore the structure that is inherited by hom-sets of the form $\mathbf{C}(A, A)$ and $\mathbf{C}(A, B)$ when $\mathbf{C}$ has biproducts.

3.1 Biproducts

Definition 3.1.1 (Zero Object)**.** A *zero object* in a category is an object O which has unique morphisms $0_A : A \to O$ and $0^A : O \to A$ for each object A. Between any two objects A and B in a category with a zero object there is a unique morphism $0_A^B : A \to O \to B$ with $0_A^B = 0^B \circ 0_A$ called the *zero morphism*.

Lemma 3.1.2. *Zero morphisms have the following properties:*

1. $0_A \circ 0^A = 1_O$.
2. *Zero objects are unique up to isomorphism.*
3. *For each* $f : A \to B$ *and each* X*, it follows that* $0_B^X \circ f = 0_A^X$ *and* $f \circ 0_X^A = 0_X^B$.

Proof.

1. $O \to A \to O = 1_O$ by uniqueness.
2. Let O and O' be zero objects then there exists unique morphisms $O \to O'$ and $O' \to O$, it follows from uniqueness that $O \to O' \to O = 1_O$ and $O' \to O \to O' = 1_{O'}$ and so $O \cong O'$.
3. Any map $A \to B$ which composes with a zero morphism factors through the zero object so $A \to B \to O \to X = A \to O \to X$ and $X \to O \to A \to B = X \to O \to B$ by uniqueness.

□

Definition 3.1.3 (Biproduct)**.** Let $\mathbf{C}$ be a category with zero objects. A *biproduct* of $\mathbf{C}$-objects A and B is a $\mathbf{C}$-object $A \oplus B$ that is both a product $(A \oplus B, p_A : A \oplus B \to$

$A, p_B : A \oplus B \to B)$ and a coproduct $(A \oplus B, i_A : A \to A \oplus B, i_B : B \to A \oplus B)$ such that,

$$p_A \circ i_A = 1_A \qquad p_A \circ i_B = 0_B^A$$
$$p_B \circ i_A = 0_A^B \qquad p_B \circ i_B = 1_B$$

A biproduct of a finite family of objects $\{A_k\}_{k \in I}$ with index set I is defined analogously. We then write, $\bigoplus_{k \in I} A_k$ with the projections and embeddings satisfying, $p_k \circ i_k = 1$ and $p_k \circ i_l = 0$ for $k \neq l$.

Given two morphisms $f : A \to C$ and $g : B \to D$, the universal properties of the product and coproduct induce unique morphisms $f \times g : A \times B \to C \times D$ and $f + g : A + B \to C + D$ respecitively.

Proposition 3.1.4. *When $A \times B$, $A + B$ and $C \times D$, $C + D$ define biproducts $A \oplus B$ and $C \oplus D$ respectively, the unique maps $f \times g$ and $f + g$ are equal.*

Proof. The morphisms $f \times g$ and $f + g$ denote the unique morphisms induced by the universal properties of the product and coproduct respectively, satisfying the commuting diagrams,

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times B \arrow[l, "p_A"'] \arrow[r, "p_B"] \arrow[d, dotted, "f \times g"] & B \arrow[d, "g"] \\
C & C \times D \arrow[l, "p_C"] \arrow[r, "p_D"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[d, "f"'] & A + B \arrow[d, dotted, "f+g"] & A \arrow[l, "i_2"'] \arrow[d, "g"] \\
C \arrow[r, "i_C"'] & C + D & D \arrow[l, "i_D"]
\end{tikzcd}$$

To see that $f \times g$ satisfies the conditions for $f + g$ we will use the fact that two morphisms $h, k : A \times B \to C \times D$ are equal when $p_C \circ h = p_C \circ k$ and $p_D \circ h = p_D \circ k$.

Following from the definition of $f \times g$ we have,

$$p_C \circ (f \times g) \circ i_A = f \circ p_A \circ i_A = f \circ \mathrm{id}_A = f = \mathrm{id}_C \circ f = p_C \circ i_C \circ f$$

and

$$p_D \circ (f \times g) \circ i_A = g \circ p_B \circ i_A = g \circ 0_A^B = 0_A^D = 0_C^D \circ f = p_D \circ i_C \circ f$$

hence $(f \times g) \circ i_A = i_C \circ f$ and similarly $(f \times g) \circ i_B = i_D \circ g$. Therefore $f \times g$ satisfies the conditions for $f + g$ and by uniqueness $f + g = f \times g$.

$\square$

Remark 3.1.5. Because of the above proposition we can use a single morphism to denote $f + g$ and $f \times g$ written as $f \oplus g : A \oplus B \to C \oplus D$ for morphisms $f : A \to C$ and $g : B \to D$.

3.2 Linear Structure

A homset $\mathbf{C}(A, B)$ of a category $\mathbf{C}$ with binary biproducts has an addition operation $+ : \mathbf{C}(A, B) \times \mathbf{C}(A, B) \to \mathbf{C}(A, B)$, where for each $f, g \in \mathbf{C}(A, B)$ we define $f + g$ as,

$$\begin{tikzcd}
A \arrow[r, "f+g"] \arrow[d, "\Delta"'] & B \\
A \oplus A \arrow[r, "f \oplus g"'] & B \oplus B \arrow[u, "\nabla"']
\end{tikzcd}$$

Where $\Delta : A \to A \oplus A$ is the diagonal map which satisfies $p \circ \Delta = 1$ and $q \circ \Delta = 1_A$ for projections p and q of $A \oplus A$. The morphism $\nabla : A \oplus A \to A$ is the codiagonal and satisfies $\nabla \circ i = 1_A$ and $\nabla \circ j = 1_A$.

Remark 3.2.1. The notation $f + g : A \to B$ denoting this addition of $f, g : A \to B$ may be confused with the notation for the induced morphism defined by the universal property for the coproduct $h + k : A + B \to C + D$ for $h : A \to C$ and $k : B \to D$. Be assured that from now on $f + g$ will always denote the addition of f and g.

Lemma 3.2.2. *The zero morphism 0_A^B acts as a unit for $(\mathbf{C}(A, B), +)$.*

Proof. Firstly, we need to show that $i_B \circ f \circ p_A = f \oplus 0_A^B$. It is enough to show that $p_B \circ (i_B \circ f \circ p_A) = p_B \circ (f \oplus 0_A^B)$ and $q_B \circ (i_B \circ f \circ p_A) = q_B \circ (f \oplus 0_A^B)$ where p_A, q_A, i_A, j_A denote the projections and injections for the biproduct of an object A with itself:

$$\begin{tikzcd}
A \arrow[r, shift left, "i_A"] & A \oplus A \arrow[l, shift left, "p_A"] \arrow[r, shift right, "q_A"'] & A \arrow[l, shift right, "j_A"']
\end{tikzcd}$$

The first equation holds due to the following commutative diagram,

$$\begin{tikzcd}
 & B \oplus B \arrow[r, "p_B"] & B & \\
A \oplus A \arrow[ur, "f \oplus 0_A^B"] \arrow[r, "p_A"'] \arrow[dr, "f \oplus 0_A^B"'] & A \arrow[ur, "f"] \arrow[r, "f"'] \arrow[dr, "0_A^B"] & B \arrow[u, "1_B"'] \arrow[r, "i_B"'] \arrow[d, "0_B^B"] & B \oplus B \arrow[ul, "p_B"'] \arrow[dl, "q_B"] \\
 & B \oplus B \arrow[r, "q_B"'] & B &
\end{tikzcd}$$

The second equation holds by replacing p_A, p_B with q_A, q_B respectively in the above diagram. This together with the definition of the diagonal map yielding $p_A \circ \Delta = 1_A$ and dually for the codiagonal map $\nabla \circ i_B = 1_B$ it follows that,

$$\begin{tikzcd}
A \arrow[r, "f+0_A^B"] \arrow[d, "\Delta"] \arrow[dd, bend right=60, "1_A"'] & B \\
A \oplus A \arrow[r, "f \oplus 0_A^B"'] \arrow[d, "p_A"'] & B \oplus B \arrow[u, "\nabla"'] \\
A \arrow[r, "f"'] & B \arrow[u, "i_B"'] \arrow[uu, bend right=60, "1_B"']
\end{tikzcd}$$

commutes. ∎

Lemma 3.2.3. *Addition in* $(\mathbf{C}(A,B),+)$ *is commutative.*

Proof. Firstly, given the biproduct of $A \oplus B$, we show that unique morphisms $c_\times : A \times B \to B \times A$ and $c_+ : A + B \to B + A$, defined respectively by commutativity of both,

$$\begin{tikzcd}
& A \times B \arrow[dl, "q_1"'] \arrow[d, dotted, "c_\times"] \arrow[dr, "p_1"] & \\
B & B \times A \arrow[l, "p_2"] \arrow[r, "q_2"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
& A + B \arrow[d, dotted, "c_+"] & \\
B \arrow[ur, "j_1"] \arrow[r, "i_2"'] & B + A & A \arrow[ul, "i_1"'] \arrow[l, "j_2"]
\end{tikzcd}$$

are equal. Following from the definition of $c_\times$, observe that,

$$p_2 \circ c_\times \circ j_1 = q_1 \circ j_1 = 1_B = p_2 \circ i_2,$$
$$q_2 \circ c_\times \circ j_1 = p_1 \circ j_1 = 0_B^A = q_2 \circ i_2$$

so that $c_\times \circ j_1 = i_2$ and similarly $c_\times \circ j_2 = i_1$. Therefore $c_\times$ satisfies the conditions for c_+ and by uniqueness $c_\times = c_+$. We will denote this map $c := c_\times = c_+$. It follows that,

$$p \circ c \circ (f \oplus g) = q \circ f \oplus g = g = p \circ g \oplus f,$$
$$q \circ c \circ (f \oplus g) = p \circ f \oplus g = f = q \circ g \oplus f$$

and hence,

$$\begin{tikzcd}
A \oplus A \arrow[r, "f \oplus g"] \arrow[d, "c"'] & B \oplus B \arrow[d, "c"] \\
A \oplus A \arrow[r, "g \oplus f"'] & B \oplus B
\end{tikzcd}$$

commutes. The diagonal map $\Delta : A \to A \oplus A$ relates to c as follows,

$$p \circ c_A \circ \Delta = q \circ \Delta = \mathrm{id}_A = p \circ \Delta,$$
$$q \circ c_A \circ \Delta = p \circ \Delta = \mathrm{id}_A = q \circ \Delta$$

and hence the triangles,

$$\begin{tikzcd}
A \arrow[r, "\Delta"'] \arrow[dr, "\Delta"'] & A \oplus A \arrow[d, "c_A"] \\
& A \oplus A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \oplus B \arrow[d, "c_B"'] \arrow[dr, "\nabla"] & \\
B \oplus B \arrow[r, "\nabla"] & B
\end{tikzcd}$$

commute, with the second triangle commuting using a dual argument for the codiagonal map. Together with the previous commuting square we have the commuting diagram,

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[dr, "\Delta"] \arrow[rrr, bend left, "f+g"] \arrow[rrrd, bend right, "g+f"'] & A \oplus A \arrow[r, "f \oplus g"] \arrow[d, "c_A"] & B \oplus B \arrow[d, "c_B"'] \arrow[dr, "\nabla"] & \\
& A \oplus A \arrow[r, "g \oplus f"'] & B \oplus B \arrow[r, "\nabla"'] & B
\end{tikzcd}$$

□

Lemma 3.2.4. *Addition in* $(\mathbf{C}(A,B),+)$ *is associative.*

Proof. Let $\alpha : (A \oplus A) \oplus A \to A \oplus (A \oplus A)$ be the associativity isomorphism for $\oplus$. It follows that both,

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \oplus A \arrow[r, "\Delta\oplus 1"] & (A \oplus A) \oplus A \arrow[d, "\alpha"] \\
A \oplus A \arrow[rr, "1\oplus\Delta"'] & & A \oplus (A \oplus A)
\end{tikzcd}
\qquad
\begin{tikzcd}
(B \oplus B) \oplus B \arrow[r, "\nabla\oplus 1"] \arrow[d, "\alpha"'] & B \oplus B \arrow[r, "\nabla"] & B \\
B \oplus (B \oplus B) \arrow[rr, "1\oplus\nabla"'] & & B \oplus B \arrow[u, "\nabla"']
\end{tikzcd}$$

commute and so the following diagram,

$$\begin{tikzcd}
A \arrow[r, "(\Delta\oplus 1)\circ\Delta"'] \arrow[rrr, bend left, "1"] \arrow[d, "(f+g)+h"'] & (A \oplus A) \oplus A \arrow[r, "\alpha"] \arrow[d, "(f\oplus g)\oplus h"'] & A \oplus (A \oplus A) \arrow[d, "f\oplus(g\oplus h)"] & A \arrow[l, "(1\oplus\Delta)\circ\Delta"] \arrow[d, "f+(g+h)"] \\
B & (B \oplus B) \oplus B \arrow[l, "\nabla\circ(\nabla\oplus 1)"'] \arrow[r, "\alpha"'] & B \oplus (B \oplus B) \arrow[r, "\nabla\circ(1\oplus\nabla)"] & B \arrow[from=2-1, to=2-4, bend right, "1"']
\end{tikzcd}$$

commutes. □

This leads us to the following result.

Theorem 3.2.5. *Let* $\mathbf{C}$ *be a category with binary biproducts. Then* $(\mathbf{C}(A,B),+,0_A^B)$ *is a commutative monoid.*

Proof. Lemmas 3.2.2, 3.2.3, 3.2.4. □

Lemma 3.2.6. *Let* $\mathbf{C}$ *be a category with binary biproducts. The action defined as,*

$$\begin{aligned}
\cdot : \mathbf{C}(A,B) \times \mathbf{C}(A,A) &\to \mathbf{C}(A,B) \\
(f,\lambda) &\mapsto f \cdot \lambda := f \circ \lambda
\end{aligned}$$

is distributive over $+$ *in* $\mathbf{C}(A,B)$*, i.e. for each* $\lambda, \mu \in \mathbf{C}(A,A)$ *and* $f, g, \in \mathbf{C}(A,B)$*,*

- $(f+g) \cdot \lambda = f \cdot \lambda + g \cdot \lambda$
- $f \cdot (\lambda + \mu) = f \cdot \lambda + f \cdot \mu$
- $f \cdot 0_A^A = 0_A^B$ *and* $0_A^B \cdot \lambda = 0_A^B$

Proof. Given $\lambda \in \mathbf{C}(A,A)$, the functoriality of Δ implies we have $\Delta \circ \lambda = \lambda \oplus \lambda \circ \Delta$. For $f, g \in \mathbf{C}(A,B)$ we have $(f \oplus g) \circ (\lambda \oplus \lambda) = (f \circ \lambda) \oplus (g \circ \lambda)$ by the UMP of $\oplus$. This allows,

$$\begin{tikzcd}
A \arrow[r, "\lambda"] \arrow[dd, "\Delta"'] & A \arrow[r, "f+g"] \arrow[d, "\Delta"] & B \\
& A \oplus A \arrow[dr, "f\oplus g"] & \\
A \oplus A \arrow[ur, "\lambda\oplus\lambda"] \arrow[rr, "(f\circ\lambda)\oplus(g\circ\lambda)"'] & & B \oplus B \arrow[uu, "\nabla"']
\end{tikzcd}$$

to commute and so $(f+g)\cdot\lambda = (f\cdot\lambda)+(g\cdot\lambda)$. In a similar way, the distribution of $f \in \mathbf{C}(A,B)$ across $+$ for $\lambda, \mu \in \mathbf{C}(A,A)$ is given by,

$$\begin{tikzcd}
A \arrow[r, "\lambda+\mu"] \arrow[dd, "\Delta"'] & A \arrow[r, "f"] & B \\
& A \oplus A \arrow[u, "\nabla"'] \arrow[dr, "f\oplus f"] & \\
A \oplus A \arrow[ur, "\lambda\oplus\mu"] \arrow[rr, "(f\circ\lambda)\oplus(f\circ\mu)"'] & & B \oplus B \arrow[uu, "\nabla"']
\end{tikzcd}$$

and so $f\cdot(\lambda+\mu) = (f\cdot\lambda)+(f\cdot\lambda)$. The zero identities result from the definition of the zero morphism. □

Theorem 3.2.7. *Let* $\mathbf{C}$ *be a category with binary biproducts. Then* $(\mathbf{C}(A,A),\circ,+)$ *is a semiring and* $(\mathbf{C}(A,B),\cdot,+)$ *is a right* $\mathbf{C}(A,A)$*-semimodule.*

Proof. Theorem 7.1.3 and Lemma 3.2.6. □

3.3 Matrix Notation

Morphsims between biproducts have an associated matrix notation. We write the projections and embeddings for a biproduct $A \oplus B$ as:

$$\begin{pmatrix}1 & 0\end{pmatrix} := p_A, \quad \begin{pmatrix}0 & 1\end{pmatrix} := p_B, \quad \begin{pmatrix}1\\0\end{pmatrix} := i_A, \quad \begin{pmatrix}0\\1\end{pmatrix} := i_B$$

Given morphisms $f : A \to C$, $g : A \to D$, $h : B \to C$ and $k : B \to D$ we write, $\begin{pmatrix}f\\g\end{pmatrix} : A \to C \oplus D$ for the unique map for which, $\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix} = f$ and $\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix} = g$, and write $\begin{pmatrix}f & h\end{pmatrix} : A \oplus B \to C$ for the unique map for which $\begin{pmatrix}f & h\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = f$. Likewise, we write $\begin{pmatrix}f & h\\g & k\end{pmatrix} : A \oplus B \to C \oplus D$ for the map that satisfies $\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}f & h\\g & k\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}f\\g\end{pmatrix} = f$ and similarly for each other component. We can then write $f \oplus g := \begin{pmatrix}f & 0\\0 & g\end{pmatrix}$. Other relevant morphisms with matrix representations are the diagonal map $\Delta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the codiagonal map $\nabla = \begin{pmatrix} 1 & 1 \end{pmatrix}$. Using the matrix notation the biproduct may be defined as the following.

Definition 3.3.1 (Biproduct)**.** Let $\mathbf{C}$ be a category with zero objects. The *biproduct* of $\mathbf{C}$-objects A and B is a $\mathbf{C}$-object $A \oplus B$ that is both a product $(A \oplus B, p_A : A \oplus B \to A, p_B : A \oplus B \to B)$ and a coproduct $(A \oplus B, i_A : A \to A \oplus B, i_B : B \to A \oplus B)$ such that the map

$$A + B \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} A \times B$$

is identity with $+$ and $\times$ representing $\oplus$ as a coproduct and product, respectively.

4

Dagger Category Theory

4.1 Dagger Categories

The term "dagger" has its roots in the notation for the adjoint of a linear operator T on a Hilbert space when used in conjunction with the bra-ket notation in quantum mechanics as in $\langle Tx \mid y \rangle = \langle x \mid T^{\dagger} y \rangle$. This notion has been abstracted using compact closed categories in the works of [1, 18]. In this setting, the dagger of a quantum process $T : H \to K$ between Hilbert spaces H and K is a reversal of this process and is represented by $T^{\dagger} : K \to H$. While the physical motivation for dagger categories is well established [1, 5, 18], there are also mathematical and categorical reasons for studying dagger categories which are explored in [12].

Definition 4.1.1 (Dagger)**.** A *dagger* on a category $\mathbf{C}$ is a contravariant involutive endofunctor that acts as the identity on objects, which is to say that a dagger is a functor $\dagger : \mathbf{C}^{op} \to \mathbf{C}$ satisfying,

- $\dagger : A \mapsto A$
- $\dagger \circ \dagger = \mathrm{id}_{\mathbf{C}}$

The dagger of a morphism $f : A \to B$ of $\mathbf{C}$ is denoted $f^{\dagger} : B \to A$.

Definition 4.1.2 (Dagger Category)**.** A *dagger category* $(\mathbf{C}, \dagger)$ is a category $\mathbf{C}$ equipped with a dagger. In other words, for each object $A, B, C \in \mathrm{ob}\mathbf{C}$ and $f \in \mathbf{C}(A, B)$ and $g \in \mathbf{C}(B, C)$.

- $\mathrm{id}_A^{\dagger} = \mathrm{id}_A$
- $(f^{\dagger})^{\dagger} = f$
- $(g \circ f)^{\dagger} = f^{\dagger} \circ g^{\dagger}$.

Some examples of dagger categories are:

- **Hilb**, the category of Hilbert spaces and bounded linear maps with dagger as the adjoints to bounded linear maps.
- **Rel**, the category of sets and relations with dagger $R^\dagger$ of a relation R is defined by $bR^\dagger a$ if and only if aRb.
- A monoid M equipped with involution $f : M^{\mathrm{op}} \to M$ where the dagger of a morphism m is $m^\dagger = f(m)$.
- Any groupoid, where the dagger is given as $f^\dagger = f^{-1}$ for a morphism f.
- **FVect**, the category of finite dimensional vector spaces and linear maps is a dagger category by choosing a basis for each object and then taking transposes for the dagger; however there is no canonical choice of dagger.

A morphism $f : H \to K$ in a dagger category can have special properties that are analogous to the familiar properties of morphisms found in categories:

(i) *dagger monomorphism* or *isometry*: $f^\dagger \circ f = \mathrm{id}_H$

(ii) *dagger epimorphism* or *coisometry*: $f \circ f^\dagger = \mathrm{id}_K$

(iii) *dagger isomorphism* or *unitary* morphism: $f^\dagger \circ f = \mathrm{id}_H$ and $f \circ f^\dagger = \mathrm{id}_K$

(iv) *idempotent*: $f \circ f = f$ with $H = K$

(v) *projection*: $f^\dagger \circ f = f$ with $H = K$

(vi) *self-adjoint:* $f^\dagger = f$ with $H = K$

Dagger categories have an inherent notion of duality which is due to the functoriality of the dagger. A simple example of this phenomenon is seen between (i) and (ii). If f is a dagger monomorphism as in (i), then $f^\dagger$ satisfies (ii) and is thus a dagger epimorphism.

Proposition 4.1.3. *The following are some simple statements about these special morphisms:*

1. *If $f : H \to K$ is a dagger monomorphism, then $f^\dagger$ is a dagger epimorphism.*
2. *$p : H \to H$ is a projection if and only if f is idempotent and self-adjoint.*

Proof. 1. This follows directly from the definition.

2. Suppose $p : H \to H$ is a projection, then $p^\dagger = (p^\dagger \circ p)^\dagger = p^\dagger \circ p^{\dagger\dagger} = p^\dagger \circ p = p$ and so p is self-adjoint, $p^\dagger \circ p = p$ implies $p \circ p = p$ and hence p is idempotent. The converse is shown by direct substitution.

□

The zero object O, if it exists, in a dagger category retains its usual properties however, the uniqueness of the morphisms $0_A : A \to O$ and $0^A : O \to A$ for an object A means that $0_A^\dagger = 0^A$.

Definition 4.1.4 (Dagger Biproduct)**.** Let $\mathbf{C}$ be a dagger category with zero object. A *dagger biproduct* of two objects H and K is a biproduct $H \oplus K$ with projections $p : H \oplus K \to H$, $q : H \oplus K \to K$ and embeddings $i : H \to H \oplus K$, $j : K \to H \oplus K$ that satisfy $p = i^\dagger$ and $q = j^\dagger$.

Proposition 4.1.5. *Let* $\mathbf{C}$ *be a dagger category with dagger biproducts. Then the dagger distributes over the* $\oplus$ *in the sense that*

$$(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$$

for each $f, g : H \to K$.

Proof. Observe that for $f \oplus g : H \oplus H \to K \oplus K$ defined as the induced map satisfying,

$$\begin{tikzcd} H \arrow[d, "f"'] & H \oplus H \arrow[l, "p_H"'] \arrow[r, "q_H"] \arrow[d, dotted, "f \oplus g"] & H \arrow[d, "g"] \\ K & K \oplus K \arrow[l, "p_K"] \arrow[r, "q_K"'] & K \end{tikzcd}$$

as in Remark 3.1.5 we have

$$\begin{aligned} p_H \circ (f \oplus g)^\dagger &= (f \oplus g \circ p_H^\dagger)^\dagger = (f \oplus g \circ i_H)^\dagger = (i_K \circ f)^\dagger = f^\dagger \circ i_K^\dagger = f^\dagger \circ p_K, \\ q_H \circ (f \oplus g)^\dagger &= (f \oplus g \circ q_H^\dagger)^\dagger = (f \oplus g \circ j_H)^\dagger = (j_K \circ f)^\dagger = f^\dagger \circ j_K^\dagger = f^\dagger \circ q_K \end{aligned}$$

and so $(f \oplus g)^\dagger$ satisfies the conditions for the canonical map $f^\dagger \oplus g^\dagger$. Therefore by uniqueness $(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger$.

$\square$

Definition 4.1.6 (Dagger Subobject)**.** Let X be an object of a category $\mathbf{C}$. A *subobject* of X is an isomorphism class of monomorphisms with codomain X. A *dagger subobject* of X is a subobject of X which contains a dagger monomorphism.

Subobjects are referred to using a representative monomorphism from the isomorphism class while dagger subobjects have a representative dagger monomorphism. Two dagger monomorphisms $a : A \hookrightarrow X$ and $b : B \hookrightarrow X$ represent the same subobject when there exists an isomorphism $h : A \to B$ with $b \circ h = a$, we then say $a = b$ *as subobjects* or by abuse of notation, $A = B$. Such an isomorphism is automatically unitary.

Definition 4.1.7 (Dagger Equaliser)**.** An *equaliser* for a pair of morphisms $f, g : H \rightrightarrows K$ is a morphism $e : E \to H$ such that $f \circ e = g \circ e$ and for any morphism $x : X \to H$ that satisfies $f \circ x = g \circ x$ there exists a unique morphism $h : X \to E$ with $x = e \circ h$. A *dagger equaliser* is an equaliser which is also a dagger monomorphism. The dual notion is a dagger coequaliser.

Proposition 4.1.8. *If a dagger category* $\mathbf{C}$ *has dagger equalisers then it has dagger coequalisers.*

Proof. Let $e : E \to A$ be the dagger equaliser to an arbitrary pair of morphism $f, g : A \to B$. For each $x : X \to A$ that satisfies $f \circ x = g \circ x$ we have a unique morphism $h : X \to E$ such that $x = e \circ h$ i.e.

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ X \arrow[u, dotted, "h"] \arrow[ur, "x"'] & & \end{tikzcd}$$

The dagger induces the dual notion so that we have a diagram with reversed arrows,

$$\begin{tikzcd}
E \arrow[d, dotted, "h^\dagger"'] & A \arrow[l, "e^\dagger"'] \arrow[dl, "x^\dagger"] & B \arrow[l, shift right, "f^\dagger"'] \arrow[l, shift left, "g^\dagger"] \\
X
\end{tikzcd}$$

The morphism $e^\dagger$ is precisely the coequaliser of $f^\dagger$ and $g^\dagger$. In this sense, the dagger sends dagger equalisers in $\mathbf{C}$ to dagger coequalisers in $\mathbf{C}$. Thus C has dagger equalisers and dagger coequalisers. $\square$

Definition 4.1.9 (Dagger Kernel)**.** A *kernel* to a morphism $f : H \to K$ is an equaliser $\ker f : L \to H$ to the pair $f, 0_H^K : H \rightrightarrows K$. The *dagger kernel*, if it exists, is a kernel which is also a dagger monomorphism. The dagger kernel to f is denoted $\ker f$. The dual concept for the dagger kernel of f is the *dagger cokernel* which is a dagger epimorphism and is denoted $\mathrm{cok} f$.

Proposition 4.1.10. *The kernel to a zero morphism is the identity.*

Proof. Trivially, $0_B \circ \mathrm{id}_B = 0_B$. For id_B to be the kernel to 0_B we need id_B to satisfy the universal property. Since any morphism $f : A \to B$ satisfies $0_B \circ f = 0_A$, we can take f to be the unique morphism such that $f = \mathrm{id}_B \circ f$. $\square$

Proposition 4.1.11. *Let* $\mathbf{C}$ *be a dagger category with dagger kernels. Then for each morphism* f*,* $\ker(f^\dagger) = (\mathrm{cok} f)^\dagger$ *as subobjects.*

Proof. This follows directly from Proposition 4.1.8. $\square$

Proposition 4.1.12. *Let* $\mathbf{C}$ *be a dagger category with dagger equalisers and dagger biproducts. Then any morphism* f *of* C *can be written as* $f = m \circ e$ *where* m *is a dagger monomorphism and* e *is an epimorphism.*

Proof. Let $f : A \to B$ be a morphism of $\mathbf{C}$. The category $\mathbf{C}$ has pushouts since any pair of morphisms $A \leftarrow B \to C$ have a coequaliser and a coproduct by assumption. We then take $m : X \to B$ to be the dagger equaliser to the cokernel pair $f_1, f_2 : B \rightrightarrows C$ of f. Then there exists a unique morphism $e : A \to X$ such that, $f = m \circ e$. By definition, m is a dagger monomorphism. It remains to show that e is an epimorphism.

It is enough to show that the cokernel pair $e_1, e_2 : X \rightrightarrows P$ to e are equal. Take the pushouts P, Q, R and S as seen in the following diagram,

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "e"'] & X \arrow[r, "m"] \arrow[d, "e_2"] & B \arrow[d, "e_2'"] \\
X \arrow[r, "e_1"'] \arrow[d, "m"'] & P \arrow[r, "m_2"'] \arrow[d, "m_1"] & R \arrow[d, "m_1'"] \\
B \arrow[r, "e_1'"'] & Q \arrow[r, "m_2'"'] & S
\end{tikzcd}$$

Since $f = m \circ e$, it follows from uniqueness of the cokernel pair of f that $f_1 = m_1' \circ e_2'$, $f_2 = m_2' \circ e_1'$ and $S = C$. Then,

$$\begin{aligned} m_2' \circ m_1 \circ e_2 &= m_1' \circ e_2' \circ m \\ &= m_2' \circ e_1' \circ m && (m \text{ is an equaliser to } f_1, f_2) \\ &= m_2' \circ m_1 \circ e_1 \end{aligned}$$

Since m is a dagger monomorphism, stability under pushout means that m_1 and m_2' are monomorphisms, hence $m_2' \circ m_1$ is a monomorphism and so $e_2 = e_1$.

□

Remark 4.1.13. A more general result holds for when a category has pushouts, equalisers to cokernel pairs, and any pushout of a regular monomorphism is a monomorphism[1].

Definition 4.1.14 (Dagger Kernel Condition)**.** A dagger category $\mathbf{C}$ with a zero object has the *dagger kernel condition* if each dagger monomorphism $f : A \to B$ in $\mathbf{C}$ is the dagger kernel of some morphism $g : B \to C$ in C so that $f = \ker g$.

Proposition 4.1.15. *Let* $\mathbf{C}$ *be a dagger category with dagger equalisers and the dagger kernel condition. If* f *is a dagger monomorphism then* $f = \ker(\mathrm{cok} f)$ *as subobjects.*

Proof. Let f be a dagger monomorphism. By the dagger kernel condition, $f = \ker w$ for some morphism w. We construct the following diagram and the induced maps r, s and t to show $f = g$ as subobjects.

$$\begin{tikzcd} & & C \arrow[d, dotted, "r"] \\ A \arrow[r, "f := \ker w"] \arrow[d, dotted, shift left, "s"] & B \arrow[r, "w"] \arrow[ur, "\mathrm{cok}(\ker w) =: l"] & W \\ K \arrow[u, dotted, shift left, "t"] \arrow[ur, "g := \ker(\mathrm{cok}(\ker w))"'] & & \end{tikzcd}$$

$w \circ f = 0$ implies there exists a unique r with $w = r \circ l$. Then $l \circ f = 0$ implies there exists a unique s with $f = g \circ s$. Since,

$$w \circ g = (r \circ l) \circ g = r \circ \mathrm{cok}(\ker w) \circ \ker(\mathrm{cok}(\ker w)) = r \circ 0 = 0$$

there exists a unique t such that $g = f \circ t$. Since f and g are dagger monic it follows that,

$$\begin{aligned} f = g \circ s = (f \circ t) \circ s &\quad\Longrightarrow\quad 1_A = t \circ s, \\ g = f \circ t = (g \circ s) \circ t = t \circ s &\quad\Longrightarrow\quad 1_K = s \circ t. \end{aligned}$$

Therefore $f = g = \ker(\mathrm{cok}(\ker w)) = \ker(\mathrm{cok} f)$ as subobjects. □

Corollary 4.1.16. *The dual statement holds in the sense that if* f *is a dagger epimorphism then* $f = \mathrm{cok}(\ker f)$ *as subobjects.*

[1]See Chapter 2 of [3]

Proposition 4.1.17. *Let* $\mathbf{C}$ *be a dagger category with dagger equalisers and the dagger kernel condition. Then* f *is a monomorphism if and only if* $\ker f = 0$.

Proof. Suppose f is a monomorphism, then $f \circ \ker f = 0 = f \circ 0$ implies $\ker f = 0$. For the converse, suppose $\ker f = 0$. Let $f \circ g = f \circ h$ and let e be the dagger coequaliser of g and h. Then there is a unique s such that $f = s \circ e$. Since e is dagger epic the dagger kernel condition implies $e = \mathrm{cok} w$ for some w. It follows that

$$f \circ w = (s \circ e) \circ w = s \circ \mathrm{cok}(w) \circ w = s \circ 0 = 0$$

and so there exists a unique t such that $w = \ker(f) \circ t$ and since $\ker f = 0$ we have $w = 0 \circ t = 0$. This implies $e = \mathrm{cok} w = \mathrm{cok} 0$ and so e is an isomorphism and hence $e \circ g = e \circ h$ implies $g = h$. Thus f is a monomorphism. □

4.2 Scalars

We saw in Theorem 3.2.7 that equipping a category $\mathbf{C}$ with a biproduct structure made the hom-sets $\mathbf{C}(A, B)$ a semimodule over the semiring $\mathbf{C}(A, A)$. We now look at what happens when we equip the category with dagger structures.

Theorem 4.2.1. *Let* $\mathbf{C}$ *be a dagger category with dagger biproducts and dagger equalisers. Then the right* $\mathbf{C}(A, A)$*-semimodule* $(\mathbf{C}(A, B), +, \cdot, 0)$ *with involution* $\dagger$ *has a left linear sesquilinear form.*

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathbf{C}(A, B) \times \mathbf{C}(A, B) &\to \mathbf{C}(A, A) \\ (f, g) &\mapsto \langle f, g \rangle := g^\dagger \circ f \end{aligned}$$

Proof. Let $\lambda, \mu \in \mathbf{C}(A, A)$ and $f, g \in \mathbf{C}(A, B)$, Linearity in the first argument is a direct calculation,

- $\langle f \cdot \lambda, g \rangle = g^\dagger \circ (f \circ \lambda) = (g^\dagger \circ f) \circ \lambda = \langle f, g \rangle \circ \lambda$,
- $\langle f_1 + f_2, g \rangle = g^\dagger \circ (f_1 + f_2) = (g^\dagger \circ f_1) + (g^\dagger \circ f_2) = \langle f_1, g \rangle + \langle f_2, g \rangle$,

The distribution of the dagger over addition holds since,

$$(f + g)^\dagger = (\nabla \circ (f \oplus g) \circ \Delta)^\dagger = \Delta^\dagger \circ (f \oplus g)^\dagger \circ \nabla^\dagger = \nabla \circ f^\dagger \oplus g^\dagger \circ \Delta = f^\dagger + g^\dagger.$$

Conjugate symmetry follows from the contravarience of the dagger,

$$\langle f, g \rangle = g^\dagger \circ f = (f^\dagger \circ g)^\dagger = \langle g, f \rangle^\dagger.$$

□

We now introduce the necessary machinery on a chosen object G to discuss inverses for the multiplication and addition on $\mathbf{C}(G, G)$.

Definition 4.2.2 (Simple Object)**.** An object X in a category with zero object O is said to be *simple* when

- $0_X \neq 1_X$

- 0_X and 1_X are the only subobjects of X.

The subobjects of a simple object must be the isomorphism classes of either the zero morphism 0_X or the identity 1_X. The zero class consists of just the zero morphism whereas the identity class consists of all isomorphisms $f : A \to X$.

Proposition 4.2.3. *Let* $\mathbf{C}$ *be a category with a simple object* G*, dagger biproducts and dagger equalisers. Then* $\mathbf{C}(G, G)$ *is a division semiring with involution.*

The proof for the above proposition is essentially the argument for Schur's lemma.

Proof. We have already established that $(\mathbf{C}(G, G), +, \circ)$ is a semiring with involution $\lambda \mapsto \lambda^\dagger$. It remains to show that $\mathbf{C}(G, G)$ has multiplicative inverses. Let $\lambda \in \mathbf{C}(G, G)$ with $\lambda \neq 0$. Then λ factors as $\lambda = m \circ e$ where m is a dagger monomorphism and e is an epimorphism by Proposition 4.1.12. Since G is simple, m must be either 0 or an isomorphism, and since $m \circ e = \lambda \neq 0$ it follows that m must be an isomorphism. This makes λ composed of epimorphisms and so λ itself is an epimorphism and $\lambda^\dagger$ is a monomorphism. Since G is simple $\lambda^\dagger$ must be either 0 or an isomorphism, but $\lambda \neq 0$ and so $\lambda^\dagger$ is an isomorphism. Therefore λ is an isomorphism with inverse λ^{-1}. $\square$

Definition 4.2.4 (Generators)**.** A *family of generators* in a category is a collection $\mathcal{G}$ of objects with the property that when given a pair $f, g : A \rightrightarrows B$, if $f \circ h = g \circ h$ for each $h : G \to A$ for every $G \in \mathcal{G}$ then $f = g$. When $\mathcal{G}$ consists of just one object G then we say this object a *generator*.

A different perspective of the generator can be seen in the contrapositive form: If G is a generator then for $f, g : A \to B$, $f \neq g$ implies there exists a map $x : G \to A$ such that $f \circ x \neq g \circ x$. A generator may also be called a *separator*.

Proposition 4.2.5. *Let* $\mathbf{C}$ *be a dagger category with a simple generator* G*, dagger biproducts, dagger equalisers and the dagger kernel condition. Then the semiring* $\mathbf{C}(G, G)$ *is a division ring with involution* $f \mapsto f^\dagger$ *for each* $f : G \to G$*.*

The proof for the above proposition is essentially the proof for Lemma 1 [9].

Proof. Following from the previous result, all that remains to show is that $(\mathbf{C}, +, \circ)$ has additive inverses.

Let $\binom{x}{y} := \ker(\nabla) : K \to G \oplus G$. Then $x + y = \nabla \circ \binom{x}{y} = 0$. Either $\binom{x}{y} \neq 0$ or $\binom{x}{y} = 0$. We will show that additive inverses exist when the former case holds and that the latter case is impossible.

Suppose $\binom{x}{y} \neq 0$, then without loss of generality, take $x \neq 0$. The generator property of G implies that there exists a $z : G \to K$ such that $x \circ z \neq 0$, and so the inverse $(x \circ z)^{-1}$ exists as shown in the above proposition. Now,

$$\begin{aligned} 1 + (y \circ z) \cdot (x \circ z)^{-1} &= ((x \circ z) + (y \circ z)) \cdot (x \circ z)^{-1} \\ &= (((x + y) \circ z) \cdot (x \circ z)^{-1} \\ &= 0 \end{aligned}$$

and so 1 has an additive inverse and hence additive inverses exist.

If $\binom{x}{y} := \ker(\nabla) = 0$ then by Proposition 4.1.17, ∇ is a monomorphism. The definition of ∇ tells us that $\nabla \circ i = \nabla \circ j$, where i and j are the embeddings for $G \oplus G$, but then $i = j$. It follows that $f = g$ for any two maps $f, g : G \to X$ for any X and in particular, $1_G = 0_G^G$. This contradicts the simplicity of G. □

Theorem 4.2.6. *Let* $\mathbf{C}$ *be a dagger category with a simple generator* G*, dagger biproducts, dagger equalisers and the dagger kernel condition. Then* $(\mathbf{C}(G, A), \langle \cdot, \cdot \rangle)$ *is a right Hermitian space over the division ring* $\mathbf{C}(G, G)$ *where for* $f, g \in \mathbf{C}(G, A)$,

- *Addition:* $f + g := \nabla \circ (f \oplus g) \circ \Delta$
- *Scalar multiplication:* $f \cdot \lambda := f \circ \lambda$
- *Hermitian Form:* $\langle f, g \rangle := g^\dagger \circ f$

Proof. Theorems 3.2.7 & 4.2.1 and Proposition 4.2.5 □

Let $f : H \to K$ be a morphism of $\mathbf{C}$. Define the function $\mathbf{C}(G, f) : \mathbf{C}(G, H) \to \mathbf{C}(G, K)$ as sending $h : G \to H$ to $f \circ h : G \to K$.

Lemma 4.2.7. *Sending* $H \mapsto \mathbf{C}(G, H)$ *and* $f \mapsto \mathbf{C}(G, f)$ *defines a functor,*

$$\mathbf{C}(G, -) : \mathbf{C} \to \mathbf{Vect}_{\mathbf{C}(G,G)}$$

Proof. We know from Theorem 4.2.6 that $\mathbf{C}(G, H)$ is a right vector space. For $f : H \to K$, $\mathbf{C}(G, f)$ is linear since for $h, k \in \mathbf{C}(G, H)$ and $\lambda \in \mathbf{C}(G, G)$ we have,

$$\begin{aligned} \mathbf{C}(G, f)(h \cdot \lambda + k) &= f \circ (h \cdot \lambda + k) \\ &= f \circ (h \cdot \lambda) + f \circ k \\ &= (f \circ h) \cdot \lambda + f \circ k \\ &= \mathbf{C}(G, f)(h) \cdot \lambda + \mathbf{C}(G, f)(k) \end{aligned}$$

Composition is preserved for each $g : K \to L$ and $f : H \to K$ since,

$$\begin{aligned} \mathbf{C}(G, g \circ f)(h) &= (g \circ f) \circ h = g \circ (f \circ h) \\ &= \mathbf{C}(G, g)(f \circ h) \\ &= \mathbf{C}(G, g)(\mathbf{C}(G, f)(h)) \\ &= (\mathbf{C}(G, g) \circ \mathbf{C}(G, f))(h) \end{aligned}$$

and $\mathbf{C}(G, 1_H)(h) = 1_H \circ h = h$ and so $\mathbf{C}(G, 1_H) = 1_{\mathbf{C}(G,H)}$. □

5 Orthogonality

So far, we have seen that a hom-set of the form $\mathbf{C}(G, H)$ for any object H in a category $\mathbf{C}$, is a Hermitian space when $\mathbf{C}$ satisfies the following conditions:

(D) $\mathbf{C}$ is equipped with a dagger $\dagger : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$.

(G) $\mathbf{C}$ is equipped with an object G which is a simple generator.

(B) $\mathbf{C}$ has a zero object and any pair of objects $H, K \in \mathbf{C}$ has a dagger biproduct.

(E) Any pair of parallel morphisms f, g has a dagger equaliser.

(K) Any dagger monomorphism is a dagger kernel.

Our goal in this chapter is to understand the minimal structure needed on such a category for the Hermitian space $\mathbf{C}(G, H)$ to form an orthomodular space, which is to say that if $F \subseteq \mathbf{C}(G, H)$ then

$$F^{\perp\perp} = F \quad \implies \quad \mathbf{C}(G, H) = F \oplus F^{\perp}.$$

We first need to look at the structure of a system of dagger subobjects of H.

5.1 Dagger Subobjects

Recall that a dagger subobject of an object H has a representative dagger monomorphism. Denote the collection of dagger subobjects as $\mathrm{Sub}_{\dagger}(H)$. There is a partial ordering on $\mathrm{Sub}_{\dagger}(H)$ where for dagger subobjects $m : M \to H$ and $n : N \to H$ we have $m \leq n$ if and only if there exists a morphism $h : M \to N$ such that $m = n \circ h$.

$$\begin{tikzcd} & H & \\ M \arrow[ur, "m"] \arrow[rr, dotted, "h"'] & & N \arrow[ul, "n"'] \end{tikzcd}$$

Since n is a dagger monomorphism, $h = n^\dagger m$. In fact, this h is always a dagger monomorphism since $1 = m^\dagger \circ m = (n \circ h)^\dagger \circ n \circ h = h^\dagger \circ n^\dagger \circ n \circ h = h^\dagger \circ h$. To discuss a notion of orthogonality, we direct our attention to the language of lattices.

Definition 5.1.1 (Lattices)**.**

1. A *lattice* is a partially ordered set $(L, \leq)$ such that each pair of elements $a, b \in L$ have a least upper bound $a \vee b$, called the *join* of a and b, and a greatest lower bound $a \wedge b$, called the *meet* of a and b.

2. A lattice is *complete* when each subset $M \subseteq L$ has a join $\bigvee_{a \in M} a$. This is equivalent to the existence of a meet $\bigwedge_{a \in M} a$ for all M.

3. A lattice is *bounded* if it has a top element $1 := \bigvee_{a \in L} a$ and a bottom element $0 := \bigwedge_{a \in L} a$.

4. An *orthocomplemented lattice* or *ortholattice* is a bounded lattice where each $a \in L$ is equipped with an element $a^\perp$ called the orthocompletent of a which satisfies the following. For each $a, b \in L$,

   (i) If $a \leq b$ then $b^\perp \leq a^\perp$

   (ii) $a = (a^\perp)^\perp$

   (iii) $a \vee a^\perp = 1$, or equivalently[1] $a \wedge a^\perp = 0$.

5. An *orthomodular lattice* is an ortholattice which satisfies the *orthomodular law*,

$$\text{if } a \leq b \text{ then } b = a \vee (b \wedge a^\perp).$$

Lemma 5.1.2. *Dagger monomorphisms are stable under pullback.*

Proof. We use the following construction. Let m be a dagger monomorphism; then it is the dagger equaliser of 1_H and $mm^\dagger$. Let $h : N \to H$ be any morphism and let $m' : P \to N$ be the dagger equaliser of h and $mm^\dagger h$. Then there exists a unique morphism $h' : P \to M$ such that

$$\begin{tikzcd}
P \arrow[r, dotted, "h'"] \arrow[d, "m'"'] & M \arrow[d, "m"] & \\
N \arrow[r, "h"'] & H \arrow[r, shift left, "mm^\dagger"] \arrow[r, shift right, "1_H"'] & H
\end{tikzcd}$$

commutes i.e. $hm' = mh'$. To see that P is a pullback take $x : V \to N$ and $y : V \to M$ such that $my = hx$. Then $hx = my = mm^\dagger my = mm^\dagger hx$ and so there exists a unique $z : V \to P$ such that $x = m'z$. Now $my = hx = hm'z = mh'z$ which implies $y = h'z$ and so P is a pullback; in particular m is pulled back to a dagger monomorphism m'. □

Proposition 5.1.3. $\mathrm{Sub}_\dagger(H)$ *is an ortholattice with the orthocomplement to a dagger subobject* $m : M \to H$ *being* $m^\perp := \ker(m^\dagger)$.

[1] $a \vee a^\perp = 1$ is equivalent to $a \wedge a^\perp = 0$ since sending $a \mapsto a^\perp$ along with (i) establishes an order isomorphism between L and L^{op}.

Proof. We know from Proposition 4.1.11 and Lemma 4.1.15 with the fact that m is dagger mono that $(m^\perp)^\perp = \ker(\ker(m^\dagger)^\dagger) = \ker(\mathrm{cok}(m)) = m$. Now given subobjects $m : M \to H$ and $n : N \to H$ with $m \leq n$, there exists a unique $h : M \to N$ such that $m = n \circ h$. Since $0 = n^\dagger \ker(n^\dagger)$ we have,

$$0 = h^\dagger \circ n^\dagger \circ \ker(n^\dagger) = (n \circ h)^\dagger \circ \ker(n^\dagger) = m^\dagger \circ \ker(n^\dagger)$$

hence there exists a unique k such that $\ker(n^\dagger) = \ker(m^\dagger) \circ k$.

$$\begin{tikzcd}
 & & L \arrow[dl, "\ker(n^\dagger)"'] \arrow[d, dotted, "k"] \\
M \arrow[r, "m"] \arrow[d, dotted, "h"'] & H & K \arrow[l, "\ker(m^\dagger)"] \\
N \arrow[ur, "n"'] & &
\end{tikzcd}$$

Therefore $n^\perp = \ker(n^\dagger) \leq \ker(m^\dagger) = m^\perp$. This makes

$$\perp : \mathrm{Sub}_\dagger(H) \to \mathrm{Sub}_\dagger(H)^{\mathrm{op}} : m \to m^\perp$$

an order isomorphism.

The bottom of $\mathrm{Sub}_\dagger(H)$ is $0 = 0^H : O \to H$ since given any m, $0 = m \circ 0$ hence $0 \leq m$ and the top is $1 = 1_H : H \to H$ since $m = m \circ 1_H$ and hence $m \leq 1$. Now the meet of two dagger subobjects m and n is the pullback $m \wedge n : P \to H$ which we know to be isomorphic to a dagger subobject from the previous lemma and so is itself a dagger subobject.

$$\begin{tikzcd}
P \arrow[r, "n'"] \arrow[d, "m'"'] \arrow[dr, dotted, "m \wedge n"] & M \arrow[d, "m"] \\
N \arrow[r, "n"'] & H
\end{tikzcd}$$

Since $\perp$ is an order isomorphism, we have $(m \wedge n)^\perp = m^\perp \vee n^\perp$. Consider $m \wedge m^\perp$, this is the pullback

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, dotted, "m \wedge m^\perp"] & M \arrow[d, "m"] \\
K \arrow[r, "m^\perp"'] & H
\end{tikzcd}$$

with $mp = \ker(m^\dagger)q$ and so $p = m^\dagger \ker(m^\dagger)q = 0$ and similarly $q = 0$, so $m \wedge m^\perp = 0$. Therefore $\mathrm{Sub}_\dagger(H)$ is an ortholattice. $\square$

5.2 Completeness

If we want $\mathrm{Sub}_\dagger(H)$ to be complete as a lattice we need additional structure.

Definition 5.2.1 (Directed Poset)**.** A *directed set* or *upward directed set* $(\mathscr{J}, \leq)$ is a nonempty poset $\mathscr{J}$ for which pairs of elements have an upper bound. That is, for each $a, b \in \mathscr{J}$ there exists a $c \in \mathscr{J}$ such that $a \leq c$ and $b \leq c$.

Example 5.2.2. If a poset $\mathscr{J}$ has finite joins, then it is directed.

Definition 5.2.3 (Directed Colimit)**.** Let $\mathbf{C}$ be a category. A *directed colimit* is a colimit to a functor $D : \mathscr{J} \to \mathbf{C}$ where $\mathscr{J}$ is a directed poset.

Definition 5.2.4 (Colimit condition)**.** Let $\mathbf{C}$ be a dagger category. Let $\mathbf{C}_{\mathrm{dm}}$ denote the wide subcategory of dagger monomorphisms of $\mathbf{C}$. Then $\mathbf{C}$ satisfies the *colimit condition* when $\mathbf{C}_{\mathrm{dm}}$ has directed colimits.

For the rest of this chapter, the category $\mathbf{C}$ satisfies conditions[2] $(D), (G), (B), (E), (K)$ and the colimit condition. Let A be an arbitrary set and let $\mathscr{P}_{\mathrm{fin}}(A)$ be the poset of all the finite subsets of A ordered by inclusion. This has finite joins given by unions, and so is directed. A functor

$$\begin{aligned} D : \mathscr{P}_{\mathrm{fin}}(A) &\to \mathbf{C}_{\mathrm{dm}} \\ (R \subseteq S) &\mapsto (i_{R,S} : DR \hookrightarrow DS) \end{aligned}$$

is then a directed diagram in $\mathbf{C}_{\mathrm{dm}}$ and using the colimit condition we can form the colimit

$$\{i_R : DR \hookrightarrow C_A\}_{R \in \mathscr{P}_{\mathrm{fin}(A)}}$$

where C_A is the colimit object.

Proposition 5.2.5. *Let* $H \in \mathbf{C}$. *The ortholattice* $\mathrm{Sub}_\dagger(H)$ *is complete.*

Proof. Consider a collection $\{m_i : M_i \to H\}_{i \in I}$ for some index set I of dagger subobjects of H. If $R \in \mathscr{P}_{\mathrm{fin}}(I)$ then we can form $m_R : DR \to H$ where $m_R := \bigvee_{i \in R} m_i$. If $R, S \in \mathscr{P}_{\mathrm{fin}}(I)$ and $R \subseteq S$ then $m_R \leq m_S$ meaning there exists a morphism $i_{R,S} : DR \to DS$ so that

$$\begin{tikzcd} & H & \\ DR \arrow[ur, "m_R"] \arrow[rr, "i_{R,S}"'] & & DS \arrow[ul, "m_S"'] \end{tikzcd}$$

commutes. This defines a functor

$$\begin{aligned} D : \mathscr{P}_{\mathrm{fin}}(I) &\to \mathbf{C}_{\mathrm{dm}} \\ (S \subseteq R) &\mapsto i_{R,S} : DR \to DS. \end{aligned}$$

[2] Listed at the start of this chapter.

Since $\mathscr{P}_{\text{fin}}(I)$ is a directed poset, we can form the colimit $c_I : C_I \to H$ with $\{i_R : DR \to C_I\}_{R \in \mathscr{P}_{\text{fin}}(I)}$ of D in $\mathbf{C}_{\text{dm}}$ such that

$$\begin{tikzcd}
 & H & \\
 & C_I \arrow[u, "c_I"'] & \\
DR \arrow[uur, "m_R"] \arrow[ur, "i_R"'] \arrow[rr, "i_{R,S}"'] & & DS \arrow[uul, "m_S"'] \arrow[ul, "i_S"]
\end{tikzcd}$$

commutes for all $R \subseteq S \in \mathscr{P}_{\text{fin}}(I)$. This c_I is indeed the join of $\{m_i\}_{i \in I}$ since when given a dagger subobject $e : E \to H$ such that $m_i \leq e$ for each $i \in I$, there exists a morphism k_R such that $m_R = e \circ k_R$ for each $R \in \mathscr{P}_{\text{fin}}(I)$, and so there exists a unique $h : C_I \to E$ with $k_R = h \circ i_R$, hence

$$\begin{tikzcd}
 & H & \\
D(R) \arrow[ur, "m_R"] \arrow[r, "i_R"] \arrow[rr, bend right, "k_R"'] & C_I \arrow[u, "c_I"'] \arrow[r, dotted, "h"] & E \arrow[ul, "e"']
\end{tikzcd}$$

commutes and thus $c_I \leq e$ for arbitrary e and so $c_I = \bigvee_{i \in I} m_i$. $\square$

5.3 Closed Subspaces

Recall that a subspace F of a Hermitian space $\mathbf{C}(G, H)$ is said to be closed when $F^{\perp\perp} = F$. The set of all closed subspaces of $\mathbf{C}(G, H)$ is denoted

$$\mathrm{Sub}_C(\mathbf{C}(G, H)) := \{F \subseteq \mathbf{C}(G, H) \mid F^{\perp\perp} = F\}.$$

Theorem 5.3.1. *The function $\varphi : \mathrm{Sub}_\dagger(H) \to \mathrm{Sub}_C(\mathbf{C}(G, H))$ which takes $m : M \to H$ to $\varphi(m) = \{mg \mid g \in \mathbf{C}(G, M)\}$ is an isomorphism of ortholattices.*

Proof. Notice that $\varphi(m) = \{f \in \mathbf{C}(G, H) \mid f = mm^\dagger f\}$. We first show that φ preserves orthocomplements which is to say,

$$\varphi(m^\perp) := \{m^\perp g \mid g \in \mathbf{C}(G, K)\} = \{mg \mid g \in \mathbf{C}(G, M)\}^\perp =: \varphi(m)^\perp$$

Now $h \in \varphi(m)^\perp$ if any only if for any $g \in \mathbf{C}(G, M)$ we have,

$$0 = \langle mg, h \rangle = h^\dagger mg$$

which is equivalent to $h^\dagger m = 0$ due to the generator property of G, and equivalent to $m^\dagger h = 0$. This means precisely that h can be written as $h = \ker(m^\dagger)k = m^\perp k$ for a unique k, in other words, $h \in \varphi(m^\perp)$. Therefore φ preserves orthogonal complements. It follows that

$$\varphi(m)^{\perp\perp} = \varphi(m^\perp)^\perp = \varphi(m^{\perp\perp}) = \varphi(m)$$

and so $\varphi(m)$ is closed.

To see that φ preserves order, let $m : M \to H$ and $n : N \to H$ be dagger subobjects of H with $m \leq n$. Any element of $\varphi(m)$ is of the form mg for $g \in \mathbf{C}(G, M)$. Since $m = nh$ for some $h : M \to N$, we have $hg \in \mathbf{C}(G, N)$ and so $mg = nhg \in \varphi(n)$, hence $\varphi(m) \subseteq \varphi(n)$. To see that φ reflects order, let $\varphi(m) \leq \varphi(n)$. For any $f : G \to H$ we have $mm^\dagger f \in \varphi(m)$, and so $mm^\dagger f \in \varphi(n)$; thus $nn^\dagger mm^\dagger f = mm^\dagger f$. The generator property of G means that $mm^\dagger = nn^\dagger mm^\dagger$ and so $m = nn^\dagger m^\dagger$, hence $m \leq n$. Thus φ reflects order and is also injective.

To see that φ is surjective, suppose that $F \subseteq \mathbf{C}(G, H)$ with $F^{\perp\perp} = F$ and let m be the join of all $m_i : M_i \to F$ with $\varphi(m_i) \subseteq F$. We show that $F = \varphi(m)$.

(i) Let $f \in F$. By Proposition 4.1.12, f decomposes as,

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "e"'] & & H \\ & X \arrow[ur, "n"'] & \end{tikzcd}$$

where n is a dagger monomorphism and e is an epimorphism. Then let $y \in F^\perp$ so then $e^\dagger n^\dagger y = f^\dagger y = 0$, and hence $n^\dagger y = 0$. It follows for each $x : G \to X$ that $(nx)^\dagger y = x^\dagger n^\dagger y = 0$ and so $nx \in F^{\perp\perp} = F$ for each x. It follows that $\varphi(n) \subseteq F$ and hence $n \leq m$ by definition of m. Therefore $f = ne \in \varphi(m)$ and $F \subseteq \varphi(m)$.

(ii) Let $y \in F^\perp$. We first want to show that $ym^\dagger = 0$. Let y decompose as $y = ne$ with n a dagger monomorphism and e an epimorphism. Since $m_i x \in F$ for each $x : G \to M_i$ we have $y^\dagger m_i x = 0$ and by the generator property of G, $y^\dagger m_i = e^\dagger n^\dagger m_i = 0$ and since e is an epimorphism $n^\dagger m_i = 0$. It follows that $m_i = \ker(n^\dagger)h$ for some h and hence $m_i \leq n^\perp$. Since m is the join of all m_i's it follows that $m \leq n^\perp$ and hence $m \perp n$, thus $y^\dagger m = e^\dagger n^\dagger m = 0$. Now for each $g : G \to M$ we have $y^\dagger mg = 0$ and so $mg \in F^{\perp\perp}$ and therefore $\varphi(m) \subseteq F^{\perp\perp} = F$.

□

Lemma 5.3.2. *A dagger subobject $m : M \to H$ and its orthocomplement $m^\perp : M^\perp \to H$ define a dagger biproduct $H \cong M \oplus M^\perp$ with,*

$$\begin{tikzcd} M \arrow[r, shift left, "m"] & H \arrow[l, shift left, "m^\dagger"] \arrow[r, shift right, "m^{\perp\dagger}"'] & M^\perp \arrow[l, shift right, "m^\perp"'] \end{tikzcd}$$

Proof. We know that $m^\dagger m = 1$, $m^{\perp\dagger} m^\perp = 1$, $m^{\perp\dagger} m = 0$ and $m^\dagger m^\perp = 0$. All that remains is to show that $(H, m^\dagger, m^{\perp\dagger})$ is a product of M and $M^\perp$. Let X be any object and $f : X \to M$ and $g : X \to M^\perp$. Then,

$$\begin{aligned} m^\dagger(mf + m^\perp g) &= m^\dagger mf + m^\dagger m^\perp g \\ &= 1f + 0g \\ &= f \end{aligned}$$

and,

$$\begin{aligned} m^{\perp\dagger}(mf + m^{\perp}g) &= m^{\perp\dagger}mf + m^{\perp\dagger}m^{\perp}g \\ &= 0f + 1g \\ &= g \end{aligned}$$

and so $mf + m^{\perp}g : X \to H$ is a candidate for the induced map for the product of M and $M^{\perp}$.

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, dotted, "mf+m^{\perp}g"] \arrow[dr, "g"] & \\ M & H \arrow[l, "m^{\dagger}"] \arrow[r, "m^{\perp\dagger}"'] & M^{\perp} \end{tikzcd}$$

It suffices to show that $m^{\dagger}$ and $m^{\perp\dagger}$ are jointly monic. Let x and y be arbitrary morphisms with codomain H, suppose $m^{\dagger}x = m^{\dagger}y$ and $m^{\perp\dagger}x = m^{\perp\dagger}y$, then for $z = x-y$ we have $m^{\dagger}z = 0$ and $m^{\perp\dagger}z = 0$. Because $m = m^{\perp\perp} = \ker(m^{\perp\dagger})$ and $m^{\perp\dagger}z = 0$, there exists a w such that $z = mw$ and so $w = m^{\dagger}mw = m^{\dagger}z = 0$ hence $z = mw = m0 = 0$ which implies $x - y = 0$ and therefore $x = y$. Thus we have a product. Dually, the coproduct property also holds. Therefore $H \cong M \oplus M^{\perp}$. □

Theorem 5.3.3. $\mathbf{C}(G, H)$ *is an orthomodular space.*

Proof. Let F be a closed subspace of $\mathbf{C}(G, H)$. Following from Theorem 5.3.1, there exists a dagger subobject $m : M \to H$ such that $\varphi(m) = F$. Observe that for any $h \in \mathbf{C}(G, H)$, $mm^{\dagger}h \in \varphi(m)$ and $m^{\perp}m^{\perp\dagger}h \in \varphi(m^{\perp}) = \varphi(m)^{\perp}$. It follows from the previous lemma that $1_H = mm^{\dagger} + m^{\perp}m^{\perp\dagger}$ and so,

$$h = 1_H h = (mm^{\dagger} + m^{\perp}m^{\perp\dagger})h = mm^{\dagger}h + m^{\perp}m^{\perp\dagger}h$$

Thus $\mathbf{C}(G, H) = \varphi(m) \oplus \varphi(m)^{\perp}$. Therefore $\mathbf{C}(G, H)$ an is orthomodular space. □

6 Equivalence

6.1 A Characterisation of Hilb

Suppose that a category $\mathbf{C}$ satisfies the conditions:

(D) $\mathbf{C}$ is equipped with a dagger $\dagger : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$.

(G) $\mathbf{C}$ is equipped with a simple generator G.

(B) Any pair of objects $H, K \in \mathbf{C}$ has a dagger biproduct.

(E) Any pair of parallel morphisms f, g have a dagger equaliser.

(K) Any dagger monomorphism is a dagger kernel.

(C) $\mathbf{C}$ satisfies the colimit condition. (Definition 5.2.4)

We saw in chapter 5 that for any object H, the homset $\mathbf{C}(G, H)$ is an orthomodular space with scalars $\mathbf{C}(G, G)$. In this chapter we show that $\mathbf{C}(G, H)$ is a Hilbert space with scalars isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ by indirectly constructing an infinite orthonormal sequence and applying Solèr's theorem. We then establish an equivalence between the category $\mathbf{C}$ and the category of $\mathbf{C}(G, G)$-Hilbert spaces.
We will now see an application of the colimit condition, which will assist in many of the upcoming arguments. To do this we will introduce notation for the finite biproducts of an object. Let I be an arbitrary set and consider the directed poset $\mathscr{P}_{\mathrm{fin}}(I)$ of all finite subsets of I. Given any $R \in \mathscr{P}_{\mathrm{fin}(I)}$, we may take the biproduct of R copies of G, denoted $G^R := \bigoplus_R G$. Now for each $R, S \in \mathscr{P}_{\mathrm{fin}(I)}$ with $R \subseteq S$, there exists a dagger monomorphism $i_{R,S} : G^R \to G^S$ as the canonical embedding. Consider the functor,

$$D : \mathscr{P}_{\mathrm{fin}(I)} \to \mathbf{C}_{\mathrm{dm}} : (R \subseteq S) \mapsto (i_{R,S} : G^R \to G^S)$$

this is a directed diagram in $\mathbf{C}_{\mathrm{dm}}$ and so there exists a colimit $\{i_R : G^R \to G_I\}_{R \in \mathscr{P}_{\mathrm{fin}(I)}}$ in $\mathbf{C}_{\mathrm{dm}}$. Because D is a functor, composition is preserved, so if $R \subseteq S \subseteq T$ are finite subsets of I, then $i_{S,T} \circ i_{R,S} = i_{R,T}$. For a singleton $\{a\} \in \mathscr{P}_{\mathrm{fin}(I)}$ we write $i_a := i_{\{a\}}$. As

G_I is not necessarily a biproduct, we distinguish the notation of the directed colimit G_I from the biproduct G^I when I is finite. However, in a sense we may read G_I as I copies of G.

Remark 6.1.1. In the context of **Hilb**, G_I will turn out to be $l^2(I)$.

Lemma 6.1.2. *Let I be an arbitrary set. The orthomodular space $\mathbf{C}(G, G_I)$ has an orthonormal sequence indexed by I.*

Proof. For each singleton $\{a\}$ of I we have $G^{\{a\}} = \oplus_{\{a\}} G = G$ along with the canonical embeddings $i_a : G \to G_I$. We show that the set of these embeddings $\{i_a\}_{a \in I}$ form an orthonormal sequence in $\mathbf{C}(G, G_I)$.

Each i_a is a dagger monomorphism since, $\langle i_a, i_a \rangle = i_a^\dagger \circ i_a = 1$. For each $a, b \in I$ with $a \neq b$, each i_a is orthogonal to i_b since,

$$\begin{aligned}
\langle i_a, i_b \rangle &= i_b^\dagger \circ i_a \\
&= (i_{\{a,b\}} \circ i_{b,\{a,b\}})^\dagger \circ i_{\{a,b\}} \circ i_{a,\{a,b\}} \\
&= i_{b,\{a,b\}}^\dagger \circ i_{\{a,b\}}^\dagger \circ i_{\{a,b\}} \circ i_{a,\{a,b\}} \\
&= i_{b,\{a,b\}}^\dagger \circ 1 \circ i_{a,\{a,b\}} \\
&= 0
\end{aligned}$$

with the last equality being a consequence of $i_{a,\{a,b\}}$ and $i_{b,\{a,b\}}$ being the canonical embeddings of the biproduct $G^{\{a,b\}} = G^{\{a\}} \oplus G^{\{b\}}$. □

Theorem 6.1.3. *Let $H \in \mathbf{C}$. The division ring $\mathbf{C}(G, G)$ is isomorphic to $\mathbb{R}$,$\mathbb{C}$ or $\mathbb{H}$ and $\mathbf{C}(G, H)$ is a right $\mathbf{C}(G, G)$-Hilbert space.*

Proof. Following from the Lemma 6.1.2, if $I = \mathbb{N}$ then $\mathbf{C}(G, G_\mathbb{N})$ is an orthomodular space by Theorem 5.3.3 and has an infinite orthonormal system $\{i_n\}_{n \in \mathbb{N}}$, so by Solèr's Theorem 2.3.15, $\mathbf{C}(G, G_\mathbb{N})$ is a Hilbert space with scalars $\mathbf{C}(G, G)$ isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

To see that $\mathbf{C}(G, H)$ is a Hilbert space, consider the orthomodular space $\mathbf{C}(G, G_\mathbb{N} \oplus H)$. Let $i : G_\mathbb{N} \to G_\mathbb{N} \oplus H$ and $j : H \to G_\mathbb{N} \oplus H$ be the canonical embeddings for $G_\mathbb{N} \oplus H$. We then have the picture

$$\begin{tikzcd}
& \mathbf{C}(G, G_\mathbb{N} \oplus H) & \\
\mathbf{C}(G, G_\mathbb{N}) \arrow[ur, "\mathbf{C}(G,i)"] & & \mathbf{C}(G, H) \arrow[ul, "\mathbf{C}(G,j)"'] \\
& \mathbf{C}(G, G) \arrow[ul, "\mathbf{C}(G,i_n)"] &
\end{tikzcd}$$

The orthomodular space $\mathbf{C}(G, G_\mathbb{N} \oplus H)$ has an infinite sequence $\{i \circ i_{n,\mathbb{N}}\}_{n \in \mathbb{N}}$, which is orthonormal since for $n \neq m$,

$$(i \circ i_n)^\dagger \circ (i \circ i_m) = i_n^\dagger \circ i^\dagger \circ i \circ i_m = i_n^\dagger \circ i_m = 0$$

and,

$$(i \circ i_n)^\dagger \circ (i \circ i_n) = i_n^\dagger \circ i^\dagger \circ i \circ i_n = i_n^\dagger \circ i_n = 1$$

and so $\mathbf{C}(G, G_\mathbb{N} \oplus H)$ is a Hilbert space. Now consider the canonical embedding $\mathbf{C}(G, j)$ and projection $\mathbf{C}(G, j^\dagger)$ to $\mathbf{C}(G, G_\mathbb{N} \oplus H)$. Since $\mathbf{C}(G, -)$ preserves composition, it follows that $\mathbf{C}(G, jj^\dagger)$ is split idempotent,

$$\begin{tikzcd}
\mathbf{C}(G, G_\mathbb{N} \oplus H) \arrow[rr, "\mathbf{C}(G,jj^\dagger)"] \arrow[dr, "\mathbf{C}(G,j^\dagger)"'] & & \mathbf{C}(G, G_\mathbb{N} \oplus H) \\
& \mathbf{C}(G, H) \arrow[ur, "\mathbf{C}(G,j)"'] &
\end{tikzcd}$$

and so the image $\mathbf{C}(G, H)$ of $\mathbf{C}(G, jj^\dagger)$ is a Hilbert space. $\square$

6.2 Equivalence

Definition 6.2.1 (Dagger Functor)**.** A *dagger functor* is a functor $F : \mathbf{H} \to \mathbf{K}$ between dagger categories $(\mathbf{H}, \dagger)$ and $(\mathbf{K}, \dagger)$ which preserves the dagger structure i.e. for each $f \in \mathbf{H}(A, B)$,

$$F(f^\dagger) = F(f)^\dagger$$

Lemma 6.2.2. *The functor* $\mathbf{C}(G, -) : \mathbf{C} \to \mathbf{Vect}_{\mathbf{C}(G,G)}$ *lifts to a dagger functor* $\mathbf{C}(G, -) : \mathbf{C} \to \mathbf{Hilb}_{\mathbf{C}(G,G)}$.

Proof. In Lemma 4.2.7 we saw that $\mathbf{C}(G, -) : \mathbf{C} \to \mathbf{Vect}_\mathbf{C}(G, G)$ is a functor. For each $H \in \mathrm{ob}\mathbf{C}$, $\mathbf{C}(G, H)$ is a Hilbert space and so to lift the codomain of $\mathbf{C}(G, -)$ to **Hilb** we require that $\mathbf{C}(G, -)$ preserves the dagger and that $\mathbf{C}(G, f)$ is bounded for each $f : H \to K$ in $\mathbf{C}$. Let $f : H \to K$ be a morphism in $\mathbf{C}$ and then for each $h \in \mathbf{C}(G, H)$ and $k \in \mathbf{C}(G, K)$,

$$\begin{aligned}
\langle \mathbf{C}(G, f^\dagger)(k), h\rangle &= \langle f^\dagger \circ k, h\rangle \\
&= h^\dagger \circ (f^\dagger \circ k) \\
&= (h^\dagger \circ f^\dagger) \circ k \\
&= (f \circ h)^\dagger \circ k \\
&= \langle k, f \circ h\rangle \\
&= \langle k, \mathbf{C}(G, f)(h)\rangle
\end{aligned}$$

and so $\mathbf{C}(G, f^\dagger)$ is the adjoint to $\mathbf{C}(G, f)$; that is, $\mathbf{C}(G, f^\dagger) = \mathbf{C}(G, f)^\dagger$. It follows from Theorem 2.3.9 that $\mathbf{C}(G, f)$ is bounded for each $f \in \mathbf{C}$. $\square$

Definition 6.2.3 (Dagger Equivalence)**.** A *dagger equivalence* between dagger categories $(\mathbf{H}, \dagger)$ and $(\mathbf{K}, \ddagger)$ is a dagger functor $F : H \to K$ which is full, faithful, and surjective on objects up to unitary isomorphism.

The rest of this chapter will be devoted to proving that $\mathbf{C}(G, -) : \mathbf{C} \to \mathbf{Hilb}_{\mathbf{C}(G,G)}$ is a dagger equivalence. We begin with faithfulness.

Lemma 6.2.4. $\mathbf{C}(G,-) : \mathbf{C} \to \mathbf{Hilb}_{\mathbf{C}(G,G)}$ *is faithful.*

Proof. Let $f, g : H \to K$ be morphisms of $\mathbf{C}$ and suppose $\mathbf{C}(G,f) = \mathbf{C}(G,g)$. Then for each $h \in \mathbf{C}(G,H)$ we have $f \circ h = g \circ h$, so by the generator property of G we have $f = g$. □

Lemma 6.2.5. *The orthonormal system $\{i_a : G \to G_I\}_{a \in I}$ is an orthonormal basis for $\mathbf{C}(G, G_I)$.*

Proof. We prove that if $f : G \to G_I$ is orthogonal to i_a for each $a \in I$, then $f = 0$. It suffices to show that $\ker(f^\dagger)$ is an isomorphism. Suppose $f \perp i_a$, then for each $R \in \mathscr{P}_{\text{fin}}(I)$ we have,

$$f^\dagger \circ i_R \circ i_{a,R} = f^\dagger \circ i_a = 0$$

hence $f^\dagger \circ i_R = 0$ since $i_{a,R}$ is the canonical embedding for the biproduct G^R. This means that i_R is in the kernel of $f^\dagger$ and so there exists a unique $g_R : G^R \to K$ such that,

$$\begin{tikzcd}
G^R \arrow[d, dotted, "g_R"'] \arrow[dr, "i_R"] & & \\
K \arrow[r, "\ker(f^\dagger)"'] & G_I \arrow[r, "f^\dagger"'] & G
\end{tikzcd}$$

commutes. The set $\{g_R : G_R \to K\}_{R \in \mathscr{P}_{\text{fin}}(I)}$ forms a cocone over $\{i_{R,S} : G^R \to G^S\}_{R,S \in \mathscr{P}_{\text{fin}}(I)}$ since if $R \subseteq S$

$$\ker(f^\dagger) \circ g_S \circ i_{R,S} = i_S \circ i_{R,S} = i_R = \ker(f^\dagger) \circ g_R$$

and hence $g_S \circ i_{R,S} = g_R$. The universal property of the directed colimit G_I means there exists a unique $g : G_I \to K$ for which

$$\begin{tikzcd}
& K & \\
& G_I \arrow[u, dotted, "g"'] & \\
G^R \arrow[uur, "g_R"] \arrow[ur, "i_R"'] \arrow[rr, "i_{R,S}"'] & & G^S \arrow[uul, "g_S"'] \arrow[ul, "i_S"]
\end{tikzcd}$$

commutes. It follows from the previous two diagrams that

$$\begin{tikzcd}
G^R \arrow[r, "i_R"] \arrow[d, "i_R"'] \arrow[dr, "g_R"] & G_I \\
G_I \arrow[r, "g"'] & K \arrow[u, "\ker(f^\dagger)"']
\end{tikzcd}$$

commutes i.e. $i_R = \ker(f^\dagger) \circ g \circ i_R$. The universal property of G_I means $\ker(f^\dagger) \circ g = 1_{G_I}$ and thus $\ker(f^\dagger)$ is an epimorphism and thus an isomorphism. Therefore $f^\dagger = 0$ and hence $f = 0$. □

Lemma 6.2.6. $\mathbf{C}(G,-) : \mathbf{C} \to \mathbf{Hilb}_{\mathbf{C}(G,G)}$ *is surjective on objects up to unitary isomorphism.*

Proof. Let H be a Hilbert space; then from Theorem 2.3.11 there exists an orthonormal basis B for H. Following from Lemma 6.2.5, we can construct the object G_B such that $\mathbf{C}(G, G_B)$ has an orthonormal basis $\{i_a\}_{a \in B}$. Since $|\{i_a\}_{a\in B}| = |B|$, both $\mathbf{C}(G, G_B)$ and H have the same dimension. Therefore $\mathbf{C}(G, G_B) \cong H$ for each $H \in \mathbf{Hilb}$ by Theorem 2.3.13. □

Lemma 6.2.7. *Let B be an orthonormal basis for $\mathbf{C}(G,H)$. Then $H \cong G_B$.*

Proof. Since $G_B = \operatorname{colim}\{G^R \mid R \subseteq B, R \text{ finite}\}$, there exists a unique $m : G_B \to H$ in $\mathbf{C}_{\mathrm{dm}}$ such that $a = mi_a$ for each $a \in B$. We want to show that this m is a dagger epimorphism i.e. $mm^\dagger = 1$, this is equivalent to $\mathbf{C}(G, mm^\dagger) = 1$, which is equivalent to $mm^\dagger a = a$ for each $a \in B$. But $a = mi_a$ by Lemma 6.2.5 and so

$$mm^\dagger a = mm^\dagger m i_a = mi_a = a.$$

Thus m is a dagger epimorphism and therefore $H \cong G_B$. □

Lemma 6.2.8. *Let $\mathbf{C}(G,H)$ be finite dimensional. Then for each bounded linear map $T : \mathbf{C}(G,H) \to \mathbf{C}(G,K)$, there exists a morphism $t : H \to K$ such that $T = \mathbf{C}(G,t)$.*

Proof. If $\{i_a : G \to H\}_{a \in B}$ is an orthonormal basis for $\mathbf{C}(G,H)$, then the i_a exhibit H as the biproduct G^B. It follows that the $T(i_a) : G \to K$ induce a unique $t : H \to K$ with $ti_a = T(i_a)$ for each a.

$$\begin{tikzcd}
 & K \\
G \arrow[ur, "T(i_a)"] \arrow[r, "i_a"'] & H \arrow[u, dotted, "t"']
\end{tikzcd}$$

Since T is bounded and linear, it is defined by how it acts on each basis element i_a but $T(i_a) = ti_a = \mathbf{C}(G,t)(i_a)$ for each $a \in B$ and so $T = \mathbf{C}(G,t)$. □

Lemma 6.2.9. $\mathbf{C}(G,-) : \mathbf{C} \to \mathbf{Hilb}_{\mathbf{C}(G,G)}$ *is full.*

To prove that $\mathbf{C}(G,-)$ is full, we need to show that each $T : \mathbf{C}(G,H) \to \mathbf{C}(G,K)$ in $\mathbf{Hilb}_{\mathbf{C}(G,G)}$ has a corresponding $t : H \to K$ in $\mathbf{C}$. The case for when $\mathbf{C}(G,H)$ is finite dimensional has been shown by Lemma 6.2.8. The approach to proving the infinite dimensional case is as follows:

(i) Reduce to the case where $\dim \mathbf{C}(G,H) \le \dim \mathbf{C}(G,K)$.

(ii) Reduce to the case where $H = K$.

(iii) Reduce to the case where $T : \mathbf{C}(G,H) \to \mathbf{C}(G,H)$ is unitary map.

(iv) Prove that for each unitary $U : \mathbf{C}(G,H) \to \mathbf{C}(G,H)$ there exists a $t : H \to H$ such that $U = \mathbf{C}(G,t)$.

Proof. (i) If $\dim(\mathrm{dom}T) > \dim(\mathrm{cod}T)$ then consider $T^\dagger : \mathbf{C}(G,K) \to \mathbf{C}(G,H)$ which has $\dim(\mathrm{dom}T^\dagger) < \dim(\mathrm{cod}T^\dagger)$. If $T^\dagger = \mathbf{C}(G,s)$ for some $s : K \to H$, then $T = \mathbf{C}(G,s)^\dagger = \mathbf{C}(G,s^\dagger)$.

(ii) Suppose for each bounded linear map $T : \mathbf{C}(G,H) \to \mathbf{C}(G,H)$ there exists a $t : H \to K$ such that $T = \mathbf{C}(G,t)$. Then if $T : \mathbf{C}(G,H) \to \mathbf{C}(G,K)$ is a bounded linear map with $\dim \mathbf{C}(G,H) \leq \dim \mathbf{C}(G,K)$, there exist bases A and B for $\mathbf{C}(G,H)$ and $\mathbf{C}(G,K)$ respectively. It follows that $|A| \leq |B|$ and so there exists a dagger monomorphism $i_{A,B} : G_A \to G_B$, and by Lemma 6.2.7, a dagger monomorphism $m : H \to K$. It follows that $T \circ \mathbf{C}(G,m^\dagger) = \mathbf{C}(G,t)$ for some $t : K \to K$. And so

$$\begin{tikzcd} & \mathbf{C}(G,H) \arrow[r, "T"] & \mathbf{C}(G,K) \\ \mathbf{C}(G,H) \arrow[ur, "\mathbf{C}(G,1_H)"] \arrow[r, "\mathbf{C}(G,m)"'] & \mathbf{C}(G,K) \arrow[u, "\mathbf{C}(G,m^\dagger)"'] \arrow[ur, "\mathbf{C}(G,t)"'] & \end{tikzcd}$$

commutes, therefore $T = \mathbf{C}(G,tm)$.

(iii): Let $T : \mathbf{C}(G,H) \to \mathbf{C}(G,H)$ be a bounded linear map. Then by Lemma 2.3.14 there exists a family of unitary maps $\{U_1, \ldots, U_N\}$ and a family of $\mathbf{C}(G,G)$-coefficients $\{\alpha_1 \ldots \alpha_N\}$ such that $T = \alpha_1 U_1 + \cdots + \alpha_N U_N$. Let $U_i = \mathbf{C}(G,t_i)$ for some $t_i : H \to K$ for each $i \in \{1, \ldots, N\}$. Then for $t = \alpha_1 t_1 + \cdots + \alpha_N t_N$,

$$\begin{aligned} T &= \alpha_1 U_1 + \cdots + \alpha_N U_N \\ &= \alpha_1 \mathbf{C}(G,t_1) + \cdots + \alpha_N \mathbf{C}(G,t_N) \\ &= \mathbf{C}(G, \alpha_1 t_1 + \cdots + \alpha_N t_N) \\ &= \mathbf{C}(G,t). \end{aligned}$$

(iv) Let B be a basis for $\mathbf{C}(G,H)$, then by Lemma 6.2.7 it is enough to prove that for a unitary map $U : \mathbf{C}(G,G_B) \to \mathbf{C}(G,G_B)$ there exists a $t : G_B \to G_B$ such that $U = \mathbf{C}(G,t)$.

Since $\{i_a\}_{a \in B}$ is an orthonormal basis for G_B and since U is unitary, the collection of $U(i_a)$ for each $a \in B$ forms a basis in $\mathbf{C}(G,G_B)$ and moreover

$$U(i_a)^\dagger U(i_a) = \langle U(i_a), U(i_a) \rangle = \langle i_a, i_a \rangle = 1.$$

Thus $\{U(i_a)\}_{a \in B}$ is a cocone in $\mathbf{C}_{\mathrm{dm}}$. The universal property of G_B induces a unique $t : G_B \to G_B$ such that,

$$\begin{tikzcd} & G_B \\ G \arrow[ur, "U(i_a)"] \arrow[r, "i_a"'] & G_B \arrow[u, dotted, "t"'] \end{tikzcd}$$

commutes for each $a \in B$ hence $U(i_a) = \mathbf{C}(G,t)(i_a)$, and therefore $U = \mathbf{C}(G,t)$. □

Theorem 6.2.10. *The dagger functor $\mathbf{C}(G,-) : \mathbf{C} \to \mathbf{Hilb}_{\mathbf{C}(G,G)}$ is a dagger equivalence.*

Proof. This follows directly from Lemmas 6.2.4, 6.2.6, 6.2.9 □

7 Monoidal Structure

7.1 Monoidal Dagger Categories

Definition 7.1.1. A *monoidal category* $(\mathbf{C}, \otimes, I, \alpha, l, r)$ is a category $\mathbf{C}$ equipped with a functor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ called the *tensor product*, an object I called the *monoidal unit* and has natural isomorphisms,

- *associator*:
$$\alpha : (- \otimes -) \otimes - \Rightarrow - \otimes (- \otimes -)\,, \qquad \alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$
- *left unitor*:
$$l : I \otimes - \Rightarrow 1_{\mathbf{C}}\,, \qquad l_A : I \otimes A \to A$$
- *right unitor*:
$$r : 1_{\mathbf{C}} \Rightarrow - \otimes I\,, \qquad r_A : A \to A \otimes I$$

that satisfy the following triangle and pentagon identities:

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "r_A^{-1} \otimes \mathrm{id}_B"'] & & A \otimes (I \otimes B) \\
& A \otimes B \arrow[ur, "\mathrm{id}_A \otimes l_B^{-1}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (A \otimes (B \otimes C)) \otimes D \arrow[r, "\alpha_{A,B\otimes C,D}"] & A \otimes ((B \otimes C) \otimes D) \arrow[dr, "\mathrm{id}_A \otimes \alpha_{B,C,D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] \arrow[drr, "\alpha_{A\otimes B,C,D}"'] & & & A \otimes (B \otimes (C \otimes D)) \\
& & (A \otimes B) \otimes (C \otimes D) \arrow[ur, "\alpha_{A,B,C\otimes D}"'] &
\end{tikzcd}$$

Note that α, l and r also satisfy the naturality conditions where for $f : A \to A'$,$g : B \to B'$ and $h : C \to C'$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "(f\otimes g)\otimes h"] \arrow[d, "\alpha_{A,B,C}"'] & (A' \otimes B') \otimes C' \arrow[d, "\alpha_{A',B',C'}"] \\
A \otimes (B \otimes C) \arrow[r, "f\otimes(g\otimes h)"'] & A' \otimes (B' \otimes C')
\end{tikzcd}
\quad
\begin{tikzcd}
I \otimes A \arrow[r, "l_A"] \arrow[d, "\mathrm{id}_I\otimes f"'] & A \arrow[d, "f"] \\
I \otimes B \arrow[r, "l_B"] & B
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "r_A"] \arrow[d, "f"] & A \otimes I \arrow[d, "f\otimes \mathrm{id}_I"'] \\
B \arrow[r, "r_B"] & B \otimes I
\end{tikzcd}$$

The original 1963 paper *Natural Associativity and Commutativity* by Saunders Mac Lane [15] included three further conditions as axioms for a monoidal category. Max Kelly showed that they follow from the axioms stated above. One of the extra conditions is the following:

Theorem 7.1.2 (Kelly 1964 [13])**.**

$$\begin{tikzcd}
I \otimes I \arrow[d, "l_I"'] \arrow[dr, "\mathrm{id}_I"] & \\
I \arrow[r, "r_I"'] & I \otimes I
\end{tikzcd}$$

commutes.

This result is used in the following lemma.

Lemma 7.1.3. *Let* $\mathbf{C}$ *be a monoidal category, then the hom-set* $\mathbf{C}(I, I)$ *forms a commutative monoid under morphism composition and monoid unit as* id_I*.*

The following proof is known as the Eckmann-Hilton argument.

Proof. Commutativity is shown by the following commutative diagram,

$$\begin{tikzcd}
I \arrow[rr, "w"] \arrow[dr, "z"] \arrow[dd, "l_I^{-1}"', "r_I"] & & I \arrow[dr, "z"] \arrow[dd, "l_I^{-1}"', "r_I"] & \\
& I \arrow[rr, "w"] & & I \\
I \otimes I \arrow[rr, "w_I\otimes \mathrm{id}_I"] \arrow[dr, "\mathrm{id}_I\otimes z"'] & & I \otimes I \arrow[dr, "\mathrm{id}_I\otimes z"'] & \\
& I \otimes I \arrow[rr, "w_I\otimes \mathrm{id}_I"'] \arrow[uu, "l_I", "r_I^{-1}"'] & & I \otimes I \arrow[uu, "l_I", "r_I^{-1}"']
\end{tikzcd}$$

□

Definition 7.1.4 (Dagger Monoidal Category)**.** A *dagger monoidal category* is a dagger category $(\mathbf{C}, \dagger)$ with a monoidal structure $(\otimes, I, \alpha, l, r)$, such that,

- $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is a dagger functor, in other words, for each pair of morphisms $f, g \in \mathbf{C}$,

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$$

- the associator α, left unitor l and right unitor r are unitary which is to say $\alpha_{A,B,C}$, l_A and r_A are unitary isomorphisms for each $A, B, C \in \mathrm{ob}\mathbf{C}$.

Remark 7.1.5. The definition for a dagger monoidal category is also equivalent to saying that $\dagger : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ is a strict monoidal functor. In that case, the fact that the associator and unitors are unitary follows automatically.

7.2 Scalar Multiplication

In Lemma 3.2.6 we defined a scalar multiplication

$$\begin{aligned} \circ : \mathbf{C}(G, A) \times \mathbf{C}(G, G) &\to \mathbf{C}(G, A) \\ (f, \lambda) &\mapsto f \circ \lambda \end{aligned}$$

where $\circ$ is morphism composition in $\mathbf{C}$. If category $\mathbf{C}$ is equipped with monoidal structure $(\otimes, I, \alpha, l, r)$ then a scalar multiplication can be defined as

$$\begin{aligned} \bullet : \mathbf{C}(A, B) \times \mathbf{C}(I, I) &\to \mathbf{C}(I, A) \\ (f, \lambda) &\mapsto f \bullet \lambda \end{aligned}$$

where $f \bullet \lambda$ is defined as

$$\begin{tikzcd} A \arrow[r, "f\bullet\lambda"] \arrow[d, "r"'] & B \\ A \otimes I \arrow[r, "f\otimes\lambda"'] & B \otimes I \arrow[u, "r^{-1}"'] \end{tikzcd} \tag{1}$$

Theorem 7.2.1. *Let* $\mathbf{C}$ *be a monoidal category with an object* G *as the monoidal unit. Then* $\bullet = \circ$.

Proof. Let $f \in \mathbf{C}(G, A)$ and $\lambda \in \mathbf{C}(G, G)$. Then $f \circ \lambda = f \bullet \lambda$ since

$$\begin{tikzcd} & I \otimes I \arrow[rr, "f\otimes\lambda"] \arrow[dr, "\mathrm{id}\otimes\lambda"] & & A \otimes I \arrow[dr, "r_I^{-1}"] & \\ I \arrow[ur, "r_I"] \arrow[dr, "\lambda"'] & & I \otimes I \arrow[ur, "f\otimes\mathrm{id}"] \arrow[dr, "r_I^{-1}"] & & A \\ & I \arrow[ur, "r_I"] \arrow[rr, "id"'] & & I \arrow[ur, "f"'] & \end{tikzcd}$$

commutes. $\square$

7.3 Axioms for the category of Hilbert spaces

In the 2022 paper *Axioms for the category of Hilbert spaces* [9], Heunen and Kornell show that the list of axioms:

(D) $\mathbf{C}$ is equipped with a dagger $\dagger : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$.

(T) $\mathbf{C}$ is equipped with a dagger monoidal structure $\otimes$ whose unit I is a simple monoidal generator[1].

(B) Any pair of objects $H, K \in \mathbf{C}$ has a dagger biproduct.

(E) Any pair of parallel morphisms f, g have a dagger equaliser.

(K) Any dagger monomorphism is a dagger kernel.

(C) $\mathbf{C}$ satisfies the colimit condition. (Definition 5.2.4)

results in the following:

Theorem 7.3.1 (Theorem 10, [9])**.** *The dagger functor* $\mathbf{C}(I,-) : \mathbf{C} \to \mathbf{Hilb}_{\mathbf{C}(I,I)}$ *is a dagger monoidal equivalence with* $\mathbf{C}(I,I)$ *isomorphic to* $\mathbb{R}$ *or* $\mathbb{C}$.

In our characterisation, we replace (T) with

(G) $\mathbf{C}$ is equipped with a simple generator G.

which resulted in a dagger equivalence between $\mathbf{C}$ and $\mathbf{Hilb}_{\mathbf{C}(G,G)}$, with $\mathbf{C}(G,G)$ isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, from Theorem 6.2.10.

Axiom (T) implies (G), taking G to be I. In fact, in **Hilb** there is only one simple object (up to isomorphism), namely the 1-dimensional Hilbert space and so there is no real choice here. Thus the axioms from [9] are stronger than those used here, but they are also able to deduce a stronger result: a monoidal equivalence to **Hilb** rather than an equivalence. Moreover, the possible scalars for **Hilb** exclude the quaternions since $\mathbf{C}(I,I)$ will have composition equal to a tensor defined scalar multiplication by Theorem 7.2.1 and so $\mathbf{C}(I,I)$ is commutative monoid by Lemma 7.1.3. Further observation reveals that neither Theorem 6.2.10 nor Theorem 10 [9] *obviously* implies the other.

[1] A *monoidal generator* in a monoidal category is an object G with the property that when given a pair $f, g : A \otimes B \rightrightarrows C$, if $f \circ (h \otimes k) = g \circ (h \otimes k)$ for each $h : G \to A$ and each $k : G \to B$ then $f = g$.

A

Appendix A: Decomposition of Bounded Linear Operators

Lemma A.0.1. *Let $T : H \to H$ be a bounded linear map on a complex Hilbert space H. Then T is a linear combination of unitary linear maps.*

Proof. Let $T : H \to H$ be a bounded linear map. Observe that T is a linear combination of self adjoint linear maps,

$$T = \frac{1}{2}(T + T^\dagger) + \frac{1}{2i}(iT - iT^\dagger).$$

Now that we have established T as a linear combination of self adjoint maps, we show that a general self adjoint map is a linear combination of unitaries. Suppose $S' : H \to H$ is a self adjoint linear map. We can rescale S' to

$$S := \frac{S'}{2\|S'\|}$$

so that $\|S\| < 1$. Consider $1 - S^2$, this is clearly positive and self adjoint and hence so is $\sqrt{1 - S^2}$. Define $R := \sqrt{1 - S^2}$. We can then write

$$S = \frac{1}{2}(S + iR) + \frac{1}{2}(S - iR).$$

We claim that $S + iR$ and $S - iR$ are unitary linear maps. See that

$$\begin{aligned}(S + iR)(S + iR)^\dagger &= (S + iR)(S^\dagger - iR^\dagger) \\ &= (S + iR)(S - iR) \\ &= S^2 + R^2 + i(RS - SR) \\ &= 1 + i(RS - SR) \qquad (2)\end{aligned}$$

The square root lemma (p196 Theorem VI.9 [17]) tells us that since $1 - S^2$ is a positive bounded linear map, there exists a unique positive linear map B such that $B^2 = 1 - S^2$

and moreover, B commutes with each C for which commutes with $1 - S^2$. As R meets these conditions, R commutes with each commuting C of $1 - S^2$ and in particular $SR = RS$. Thus $RS - SR = 0$ and following from (2) we have

$$(S + iR)(S + iR)^\dagger = 1$$

and similarly $(S+iR)^\dagger(S+iR) = 1$ making $S+iR$ unitary. The same argument holds for $S - iR$. Therefore T is a linear combination of unitary linear maps. □

Lemma A.0.2. *Let $T : H \to H$ be a bounded linear map on an infinite dimensional real Hilbert space H. Then T is a linear combination of orthogonal (unitary) maps.*

Proof. Let $T' : H \to H$ be a bounded linear map on a separable Hilbert space H. Rescale T' so that

$$T = \frac{T'}{2\|T'\|}$$

By Lemma 3.4 [4] T can be written in the form

$$T = I - US$$

where U is orthogonal (unitary) and S is symmetric (Hermitian). Using Lemma 3.1 [4] we can write H as the orthogonal sum of two copies of an infinite dimensional S-invariant closed linear subspace of H which we'll denote as H_1 so that

$$H = H_1 \oplus H_1$$

and so S can be written as the direct sum[1],

$$S = S_1 \oplus S_2 : H_1 \oplus H_1 \to H_1 \oplus H_1$$

As in Lemma 4.2 [4] we can write $S_1 \oplus S_2$ as

$$\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} = \begin{pmatrix} \frac{S_1+S_2}{2} & 0 \\ 0 & \frac{S_1+S_2}{2} \end{pmatrix} + \begin{pmatrix} \frac{S_1-S_2}{2} & 0 \\ 0 & -\frac{S_1-S_2}{2} \end{pmatrix}. \tag{3}$$

Setting,

$$A := \frac{S_1 + S_2}{2}, \qquad B := \frac{S_1 - S_2}{2}$$

it follows as in Lemma 4.1 [4] that the first term in (3) can be written as

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A & \sqrt{I - A^2} \\ -\sqrt{I - A^2} & A \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A & -\sqrt{I - A^2} \\ \sqrt{I - A^2} & A \end{pmatrix}$$

and the second term in (3) as

$$\begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} = \frac{1}{2}\begin{pmatrix} B & \sqrt{I - B^2} \\ \sqrt{I - B^2} & -B \end{pmatrix} + \frac{1}{2}\begin{pmatrix} B & -\sqrt{I - B^2} \\ -\sqrt{I - B^2} & -B \end{pmatrix}$$

which are linear combinations of orthogonal operators and thus so is S and hence so is $T = I - US$. Therefore T' is a linear combinations of orthogonal operators. □

[1] As pointed out by Halmos in the proof of Lemma 3 [6] *"The underlying Hilbert space, if it is not already separable, can be expressed as a direct sum of separable, infinite dimensional subspaces invariant under the given operator. There is, therefore, no loss of generality in restricting attention to separable Hilbert spaces."*

Lemma A.0.3. *Let $T : H \to H$ be a bounded linear map on an infinite-dimensional quarternionic Hilbert space H. Then T is a linear combination of orthogonal (unitary) maps.*

Consider a quaternionic Hilbert space H as a real Hilbert space equipped with operators $R_i, R_j : H \to H$ which satisfy $R_i^2 = R_j^2 = -\mathrm{id}_H$, $R_iR_j = -R_jR_i$ and $R_i^\dagger = -R_i$ and $R_j^\dagger = -R_j$ as in Example 2.2.7. The $\mathbb{H}$-linear operators on H are $\mathbb{R}$-linear operators which commute with R_i and R_j. The proof for the quaternionic case is then analogous to the real case but requires each operator to commute with R_i and R_j.

Proof. Let T' be a bounded $\mathbb{R}$-linear map. Rescale T' so that

$$T = \frac{T'}{2\|T'\|}.$$

Let R commute with T'. Then

$$RT = \frac{RT'}{2\|T'\|} = \frac{T'R}{2\|T'\|} = TR.$$

Let $Q := I - T$, then $RQ = QR$ and $RQ^* = Q^*R$. Now Q is invertible since $\|T\| = 1/2 < 1$ and so following from Lemma 3.3 [4] there exists an orthogonal (unitary) operator U such that

$$Q = U|Q|$$

where $|Q| := \sqrt{Q^*Q}$, a symmetric (Hermitian) operator. By the square root lemma $R|Q| = |Q|R$ since R commutes with Q^*Q. It follows that

$$UR = Q|Q|^{-1}R = QR|Q|^{-1} = RQ|Q|^{-1} = RU.$$

Let $S := |Q|$ then S is self adjoint since,

$$(S^*)^2 = S^*S^* = (S^2)^* = (Q^*Q)^* = Q^*Q^{**} = Q^*Q = S^2$$

hence $S^* = S$. Now,

$$I - T = US \qquad \Longrightarrow \qquad T = I - US.$$

We have established that R commutes with U and S. Now we want to show that such an S is a linear combination of orthogonal operators that also commute with R. We can use use the same argument as in the proof of Lemma A.0.2 to construct such a decomposition for S, noting that the construction of Lemma 3.1 in [4] gives a decomposition which is invariant not just under S but under any operator R that commutes with S. $\square$

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