The Uniqueness of Natural Numbers

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One fish, two fish, red fish, blue fish [1, p.1]. There is something odd about this sequence as we read it in our heads. Beginning with *one* fish and then following with *two* fish feels natural, since if you see one fish and then you see another fish, you've seen two fish. However the ordering of *red* fish and then *blue* fish seems arbitrary[1], Dr Seuss may as well have said *blue* fish, *red* fish and no one would have batted an eye. What would be strange is to say you've seen two fish and then note you've seen one fish. This is given in the statement that you've seen two fish, since $2 = 1 + 1$.

Natural numbers have properties that may feel obvious or trivial but we will see that this collection of properties can be uniquely characterised, up to isomorphism, using a few primitive notions. At first I will discuss the characterisation of the natural numbers using the Peano axioms and introduce the recursion theorem. Then a recursive definition of the arithmetic of the natural numbers will be given. Then I will use the recursion theorem and inductive methods to prove that any model $(N, s', 0')$ of the Peano axioms will be structurally isomorphic to the natural numbers.

In Turin, Italy 1889, the mathematician Guiseppe Peano developed a way to describe the arithmetic of numbers using the primitive notions 0, number and 'successor of'. He used the following five postulates;

(P1) 0 is a number.
*(P2) If **a** is a number then the successor of **a** is a number.*
(P3) 0 is not the successor of any number.
(P4) If two numbers have the same successor, the two numbers are identical.
(P5) If s be a class to which belongs 0 and also the successor of every number belonging to s, then every number belongs to s.

[1] Perhaps the red fish was easier to spot than the blue fish.

We can restate these postulates as

$$\begin{array}{ll} \text{(P1)} & 0 \in \mathbb{N} \\ \text{(P2)} & n \in \mathbb{N} \implies s(n) \in \mathbb{N} \\ \text{(P3)} & n \in \mathbb{N} \implies s(n) \neq 0 \\ \text{(P4)} & s(m) = s(n) \implies m = n, \quad \forall\, m, n \in \mathbb{N} \\ \text{(P5)} & \text{Let } X \subseteq \mathbb{N} \text{ with } 0 \in X \text{ and } n \in X \implies s(n) \in X. \text{ Then } X = \mathbb{N}. \end{array}$$

It should be noted that Peano himself did not include 0 in the set of natural numbers and instead used 1 as the initial number. Hopefully by the end of this report it should be clear that these two conventions are equivalent to one another in that they both satisfy the Peano axioms. In fact we could start at any integer. This is alluded to in a quote by Peano in Russell [2 p.126].

> *All the systems which satisfy the primitive propositions have a one-one correspondence with the numbers. Number is what is obtained from all these systems by abstraction; in otherwords, number is the system which has all the properties enunciated in the primitive propositions, and only those.*

Our task will be to find this so called one-one correspondence in the form of a bijection which preserves the arithmetic of these numbers and to do this we will introduce the recursion theorem.

Theorem 1. *(Recursion Theorem)* If X is a set, $f : X \to X$ a function and $c \in X$, then there exists a unique function $\phi : \mathbb{N} \to X$ such that

$$\begin{array}{ll} \text{(R1)} & \phi(0) = c \\ \text{(R2)} & \phi(s(n)) = f(\phi(n)) \end{array}$$

The following proof is based on one by Steward and Tall [3, p.163].

Proof. Firstly we want to show that such a function ϕ exists. Let ϕ be the intersection of all subsets $U \subset \mathbb{N} \times X$ which satisfy,

$$\begin{array}{ll} \text{(R1*)} & (0, c) \in U \\ \text{(R2*)} & (n, x) \in U \implies (s(n), f(x)) \in U \end{array}$$

Now consider the set

$$S := \{n \in \mathbb{N} \mid (n, x) \in \phi \text{ for some } x \in X\}$$

By (R1*), $0 \in S$ and by (R2*) we have $n \in S \implies s(n) \in S$. Since $S \subset \mathbb{N}$, by (P5) $S = \mathbb{N}$.

So each $n \in \mathbb{N}$ is associated with some $x \in X$ such that $(n, x) \in \phi$ but we still haven't descirbed a function since we could be mapping n to multiple $x \in X$. We need to prove that x is unique.

Consider the set

$$T := \{n \in \mathbb{N} \mid (n, x) \in \phi \text{ for a unique } x \in X\}$$

We will show that $T = \mathbb{N}$ by induction.

Initial Step: Beginning with $n = 0$, we know that $(0, c) \in \phi$. If also $(0, d) \in \phi$ with $c \neq d$, let

$$\phi^- := \phi \backslash \{(0, d)\}$$

Then ϕ^- satisfies $(R1^*)$, and if $(n, x) \in \phi^-$ then $(s(n), f(x)) \in \phi$ and not $(0, d)$ because $s(n) \neq 0$ by (P3). So $(s(n), f(x)) \in \phi^-$ and ϕ^- satisfies (R2*).

But ϕ is the intersection of all subsets satisfying (R1*) and (R2*) and so ϕ^- must contain $(d, 0)$. Thus we have a contraction and so no such d can exist. Therefore $0 \in T$.

Induction step: We now need to show that $n \in T$ implies $s(n) \in T$.

If $n \in T$ then $(n, x) \in \phi$ for exactly one $x \in X$. From (R2*) we see that $(s(n), f(x)) \in \phi$, so to be sure that $s(n) \in T$ we need to show that there can be no other $(s(n), y) \in \phi$ with $y \neq f(x)$.

For the sake of contradiction, suppose there was such a $(s(n), y) \in \phi$. Consider

$$\phi^* := \phi \backslash \{(s(n), y)\}$$

Since $0 \neq s(n)$, we know that ϕ^* satisfies (R1*).

To show that (R2*) holds for ϕ^* we need to show that

$$(m, z) \in \phi^* \implies (s(m), f(z)) \in \phi^*, \quad \forall m \in \mathbb{N}$$

This is true for for $m = n$, since there is a unique $x \in X$ such that $(n, x) \in \phi$ and for this x, $(s(n), f(z)) \in \phi$ and is not $(s(n), y)$ since $y \neq f(x)$.

For $m \neq n$ we have $(s(m), f(z)) \in \phi$ and $s(m) \neq s(n)$ by $P4$. And so $(s(m), f(z)) \neq (s(n), y)$, hence $(s(m), f(z)) \in \phi^*$.

Thus ϕ^* satisfies (R2*) and since ϕ is the smallest set satisfying (R1*) and (R2*), ϕ^* must contain $(s(n), y)$ and we have a contradiction. By (P5), $T = \mathbb{N}$.

Therefore ϕ is a unique function satisfying $(R1)$ and $(R2)$.

□

Addition and Multiplication

We can use the recursion theorem to define the arithmetic of $\mathbb{N}$. So that addition is defined as a function $\alpha_m : \mathbb{N} \to \mathbb{N}$ and $\alpha_m(n) = m + n$ with

(A1) $\alpha_m(0) = m$
(A2) $\alpha_m(s(n)) = s(m + n)$

We can see how (R1) & (R2) relate to (A1) & (A2) with $c = m$ and $f(x) = s(n)$. Multiplication is defined as a function $\mu_m : \mathbb{N} \to \mathbb{N}$ and $\mu_m(n) = mn$ with

(M1) $\mu_m(0) = 0$
(M2) $\mu_m(s(n)) = \mu_m(n) + m$

relating to (R1) & (R2) with $c = 0$ and $f(r) = r + m$. It is interesting to note how multiplication (M2) is defined with addition.

$$\mu_m(s(n)) = \mu_m(n) + m = \alpha_{\mu_m(n)}(m)$$

Order $\leq$ in $\mathbb{N}$ is defined using addition so that for all $m, n \in \mathbb{N}$ as

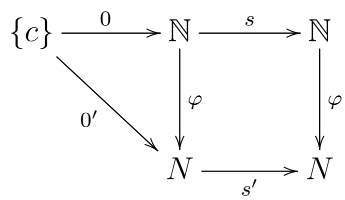
(O1) $m = n + p \implies n \leq m$

for some $p \in \mathbb{N}$.

The proofs for the standard properties required for natural number operations of these definitions can be found in [2, p.165].

Uniqueness

Uniqueness of the natural numbers for us means that we can take a set N with function $s' : N \to N$ and $0' \in N$ which satisfies (P1)-(P5) and show that $(N, s', 0')$ and $(\mathbb{N}, s', 0')$ are the same up to isomorphism. We want to find a function $\varphi : \mathbb{N} \to N$ such that the diagram



commutes, which is to say

$$\varphi(0) = 0' \tag{$B1$}$$
$$\varphi(s(n)) = s'(\varphi(n)) \tag{$B2$}$$

In the diagram, the 0 and $0'$ arrows represent a mapping of some single element set {c} to the elements $0 \in \mathbb{N}$ and $0' \in N$. This is an equivalent way to express the inclusion of an initial element.

We know such a function exists due to the recursion theorem. This also means that there is a function $\psi : N \to \mathbb{N}$ with

$$\psi(0') = 0 \tag{$B'1$}$$
$$\psi(s'(n)) = s(\psi(n)) \tag{$B'2$}$$

We can show that φ and ψ are each others inverses by induction.

Let $S := \{n \in \mathbb{N} \mid \varphi(\psi(n)) = n\}$. By definition of φ and ψ

$$\psi(\varphi(0)) = \psi(0') = 0$$

and so $0 \in S$. Now suppose $k \in S$, then

$$\begin{aligned} \psi(\varphi(s(k))) &= \psi(s'(\varphi(k))) && (B2) \\ &= s(\psi(\varphi(k))) && (B'2) \\ &= s(k) && (k \in S) \end{aligned}$$

and so $s(k) \in S$ and by (P5) $S = \mathbb{N}$. Hence $\varphi(\psi(n)) = n$ for each $n \in \mathbb{N}$. We can make a similar argument to show $\varphi(\psi(n')) = n'$ for each $n' \in N$.

Therefore $\varphi : \mathbb{N} \to N$ is a bijection.

For φ to be an isomorphism we need to be sure that addition, multiplication and order are preserved. The addition, multiplication and order in N will be denoted $\alpha'_m(n) = m +_N n$ and $\mu'_m(n) = m \cdot_N n$ and $\leq_N$ with,

(A′1) $\alpha'_m(0') = m$
(A′2) $\alpha'_m(s'(n)) = s'(m +_N n)$

(M′1) $\mu'_m(0') = 0'$
(M′2) $\mu'_m(s'(n)) = \mu_m(n) +_N m$

(O′1) $m = n + p \implies n \leq m$

Proposition 1. *Addition is preserved under φ i.e.*

$$\varphi(m + n) = \varphi(m) +_N \varphi(n)$$

Proof. For the sake of induction, let

$$S = \{n \in \mathbb{N} \mid \varphi(m + n) = \varphi(m) +_N \varphi(n)\,, m \in \mathbb{N}\}$$

Then

$$\begin{aligned}
\varphi(m + 0) &= \varphi(\alpha_m(0)) && \text{(def of } +) \\
&= \varphi(m) && (A1) \\
&= \alpha'_{\varphi(m)}(0') && (A'1) \\
&= \alpha'_{\varphi(m)}(\varphi(0)) && (A) \\
&= \varphi(m) +_N \varphi(0) && \text{(def of } +_N)
\end{aligned}$$

Hence $0 \in S$. Now suppose $k \in S$, then

$$\begin{aligned}
\varphi(m+s(k)) &= \varphi(\alpha_m(s(k))) && \text{(def of }+\text{)}\\
&= \varphi(s(\alpha_m(k))) && (A1)\\
&= s'(\varphi(\alpha_m(k))) && (B2)\\
&= s'(\varphi(m+k)) && \text{(def of }+_N\text{)}\\
&= s'(\varphi(m) +_N \varphi(k)) && (k \in S)\\
&= \varphi(m) +_N s'(\varphi(k)) && (A'2)\\
&= \varphi(m) +_N \varphi(s(k)) && (B2)
\end{aligned}$$

Hence $s(k) \in S$. By (P5) it follows that $S = \mathbb{N}$.
Therefore addition is preserved under φ.

□

The fact about φ allows us to make the corresponding statement for multiplication.

Proposition 2. *Multiplication is preserved under* φ *i.e.*

$$\varphi(mn) = \varphi(m) \cdot_N \varphi(n)$$

Proof. For the sake of induction, let

$$S = \{n \in \mathbb{N} \mid \varphi(mn) = \varphi(m) \cdot_N \varphi(n)\,, m \in \mathbb{N}\}$$

Then

$$\begin{aligned}
\varphi(m0) &= \varphi(\mu_m(0)) && \text{(def of }\cdot\text{)}\\
&= \varphi(0) && (M1)\\
&= 0' && (B1)\\
&= \mu'_{\varphi(m))}(0') && (M'1)\\
&= \mu'_{\varphi(m))}(\varphi(0)) && (B1)\\
&= \varphi(m) \cdot_N \varphi(0) && \text{(def of }\cdot_N\text{)}
\end{aligned}$$

Hence $0 \in S$. Now suppose $k \in S$, then

$$\begin{aligned}
\varphi(s(k)) &= \varphi(\mu_m(s(k))) && (\text{def of } \cdot) \\
&= \varphi(\mu_m(k) + m) && (M2) \\
&= \varphi(\mu_m(k)) +_N \varphi(m) && (\text{proposition 1}) \\
&= \varphi(mk) +_N \varphi(m) && (\text{def of } \cdot) \\
&= \varphi(m) \cdot_N \varphi(k) +_N \varphi(m) && (k \in S) \\
&= \mu'_{\varphi(m)}(\varphi(k)) +_N \varphi(m) && (\text{def of } \cdot_N) \\
&= \mu'_{\varphi(m)}(s'(\varphi(k))) && (M'2) \\
&= \mu'_{\varphi(m)}(\varphi(s(k))) && (B2) \\
&= \varphi(m) \cdot_N \varphi(s(k)) && (\text{def of } \cdot_N)
\end{aligned}$$

Hence $s(k) \in S$. By (P5) it follows that $S = \mathbb{N}$.
Therefore multiplication is preserved under φ. □

We can also show that order is preserved. This is a result from proposition 1.

Proposition 3. *Order is preserved under φ i.e.*

$$n \leq m \implies \varphi(n) \leq_N \varphi(m)$$

Proof. Suppose for some $n, m \in \mathbb{N}$, $n \leq m$. Then there is some $p \in \mathbb{N}$ such that $n + p = m$. Using proposition 1, we can see that

$$\begin{aligned}
n + p &= m \\
\varphi(n + p) &= \varphi(m) \\
\varphi(n) +_N \varphi(p) &= \varphi(m)
\end{aligned}$$

$$\implies \varphi(n) \leq_N \varphi(m)$$

so order is preserved under φ. □

We've now seen that $\varphi : \mathbb{N} \to N$ is a bijection which preserves addition, multiplication and order between the natural numbers $(\mathbb{N}, 0, s)$ and any model of the Peano axioms $(N, 0', s')$. We used $(N, 0', s')$ for could have used for $(\mathbb{N}, 1, s)$ like Peano's original axioms or even $(\mathbb{N}, 42, s)$. The recursion theorem always makes sure there exists a unique isomorphism between any two models of the Peano axioms.

1 References

[1] Geisel, Theodor Seuss. One Fish, Two Fish, Red Fish, Blue Fish. Beginner Books [U.A.], 1992.

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[4] Stoll, Robert R. Set Theory And Logic. Dover Publications, 2017.